Calibration of Photon Counting Detectors

Astronomical Photometry by Photons

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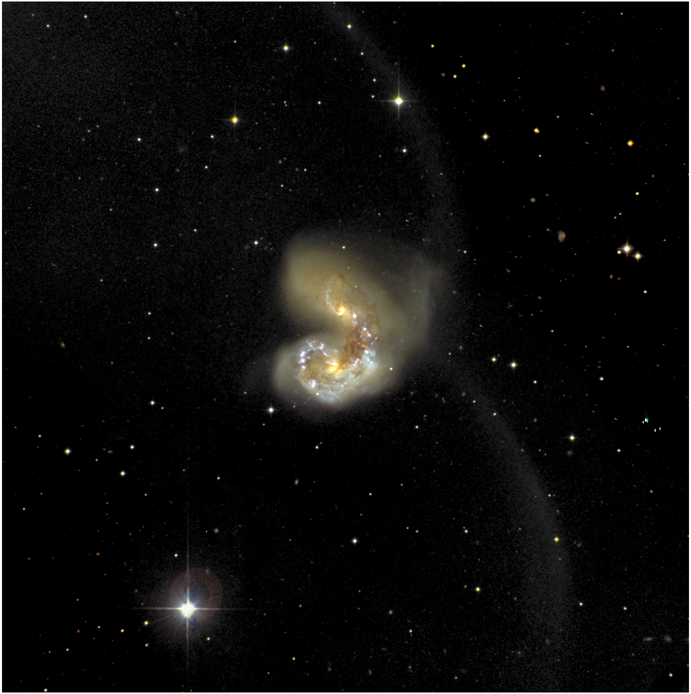


Foundations

Description of the Electromagnetic Field[1]

Maxwell's picture

$$\nabla \cdot \mathbf{E} = 0,$$
$$\nabla \cdot \mathbf{B} = 0,$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$
$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Vector potential picture

$$\mathbf{B} = \nabla \times \mathbf{A},$$
$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t},$$
$$\nabla \cdot \mathbf{A} = 0 \text{ (calibration)}.$$

$$\boxed{\nabla^2 \mathbf{A} = \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2}}$$

- Source-free field – light propagating in vacuum ($\varrho = 0, \varphi = 0$).

[1] Walls, Milburn: Quantum Optics (2008)

Quantisation of the Electromagnetic Field

Plane-wave solution ($\mathbf{A} \to \hat{\mathbf{A}}$):

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left[\hat{a}_k \mathbf{u}_k(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\,\omega_k t} + \hat{a}_k^+ \mathbf{u}_k^*(\mathbf{r})\, \mathrm{e}^{\mathrm{i}\,\omega_k t} \right],$$

$$\hat{\mathbf{E}}(\mathbf{r},t) = \mathrm{i} \sum_k \sqrt{\frac{\hbar \omega_k}{2\epsilon_0}} \left[\hat{a}_k \mathbf{u}_k(\mathbf{r})\, \mathrm{e}^{-\mathrm{i}\,\omega_k t} - \hat{a}_k^+ \mathbf{u}_k^*(\mathbf{r})\, \mathrm{e}^{\mathrm{i}\,\omega_k t} \right]$$

with quantisation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^+, \hat{a}_{k'}^+] = 0,\ [\hat{a}_k, \hat{a}_{k'}^+] = \delta_{kk'}.$$

$\hat{a}_k, \hat{a}_{k'}^+$ are creation and annihilation operators and

$$\mathbf{u}_k(\mathbf{r}) = \mathbf{e}_\lambda\, \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}}$$

the polarization and rectangular coordinates inside unit volume.

Energy of the Electromagnetic Field

Energy density $\hat{H}$ in unit volume

$$\hat{H} = \frac{1}{2} \int \left(\epsilon_0 \hat{\mathbf{E}}^2 + \frac{\hat{\mathbf{B}}^2}{\mu_0} \right) \mathrm{d}V$$

and commutation relations gives

$$\hat{H} = \sum_{k} \hbar \omega_k \left(\hat{a}_k^+ \hat{a}_k + \frac{1}{2} \right) .$$

Representation by Number States

Eigenvalues of Hamiltonian ($n_k = 0, 1, \ldots$)

$$\hat{H}|n_k\rangle = \hbar\omega_k(n_k + 1/2)|n_k\rangle$$

and Number operator $\hat{n}_k = \hat{a}_k^+ \hat{a}_k$

$$\hat{a}_k^+ \hat{a}_k |n_k\rangle = n_k |n_k\rangle.$$

Application of creation and annihilation operators

$$\hat{a}_k|n_k\rangle = \sqrt{n_k}|n_k - 1\rangle,\ \hat{a}_k^+|n_k\rangle = \sqrt{n_k + 1}|n_k + 1\rangle.$$

The states are orthogonal

$$\langle n_k|m_k\rangle = \delta_{mn}$$

and complete

$$\sum_{n_k=0}^{\infty} |n_k\rangle\langle n_k| = 1.$$

Probability of Photon Detection

Probability of detection of a single state

$$w_{if} = |\langle f|\hat{E}^{(+)}(\mathbf{r},t)|i\rangle|^2.$$

Intensity of pure state $|i\rangle$:

$$I(\mathbf{r},t) = \sum_f w_{if} = \sum_f \langle i|\hat{E}^{(-)}(\mathbf{r},t)|f\rangle\langle f|\hat{E}^{(+)}(\mathbf{r},t)|i\rangle$$
$$= \langle i|\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)|i\rangle$$

with

$$\sum_f |f\rangle\langle f| = 1.$$

Intensity of source with $P_i|i\rangle$:

$$I(\mathbf{r},t) = \sum_i P_i\langle i|\hat{E}^{(-)}(\mathbf{r},t)\hat{E}^{(+)}(\mathbf{r},t)|i\rangle.$$

Density Operator for Photon Field

Density operator for field:

$$\hat{\varrho} = \sum_i P_i |i\rangle\langle i|$$

gives intensity

$$I(\mathbf{r}, t) = \mathrm{Tr}\left\{\hat{\varrho}\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)\right\}.$$

Application on vacuum state

$$\hat{\varrho} = |0\rangle\langle 0|$$

gives

$$I(\mathbf{r}, t) = \langle 0|\hat{E}^{(-)}(\mathbf{r}, t)\hat{E}^{(+)}(\mathbf{r}, t)|0\rangle = 0.$$

Density Operator of Photon Detection[2]

- States of photon field $|n_k\rangle$
- States of detector $\tau_l|t_l\rangle$

Density operator for field and detector:

$$\hat{\varrho} = \sum_{k,l} \tau_l^* |n_k, t_l\rangle\langle t_l, n_k| \tau_l .$$

Truly detected intensity

$$I(\mathbf{r}, t) = \sum_i \tau_l \tau_l^* \langle t_l, n_k | \hat{E}^{(-)}(\mathbf{r}, t) \hat{E}^{(+)}(\mathbf{r}, t) | n_k, t_l \rangle .$$

For plane wave, $\omega = ck$, $t = \tau_l \tau_l^*$ and continuous operators

$$I(\mathbf{r}, t) = \int t(\omega) n(\omega) \hbar\omega \, \mathrm{d}\omega$$

[2]This interpretation may be completely wrong (developed by me).

Fluxes in Astronomy

Macroscopic Description of Light Flux

(Energy) flux (primary quantity) as $I\Delta\Omega/AT$ (direction along to $\mathbf{k}$):

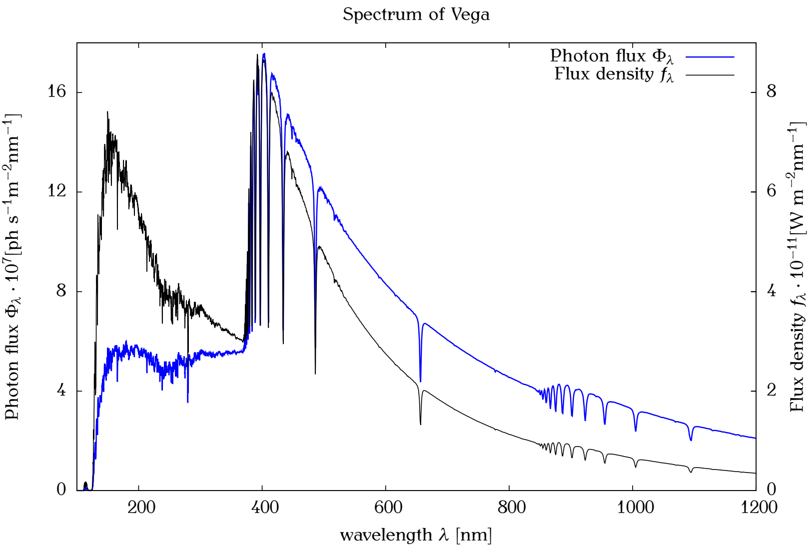
$$F = \int_0^\infty \Phi_\lambda(\lambda) t(\lambda) \frac{hc}{\lambda}\, \mathrm{d}\lambda = \int_0^\infty f_\lambda(\lambda)\, t(\lambda)\, \mathrm{d}\lambda \quad [\mathrm{W\,m^{-2}}]$$

Derived:

- Photon flux density: $\Phi_\lambda(\lambda) = n_\lambda(\lambda)/AT$ $[\mathrm{s^{-1}\,m^{-2}\,nm^{-1}}]$
- Flux density : f_λ $[\mathrm{W\,m^{-2}\,nm^{-1}}]$
- Photon flux

$$\Phi = \int_0^\infty \Phi_\lambda(\lambda) t(\lambda)\, \mathrm{d}\lambda \quad [\mathrm{s^{-1}\,m^{-2}}]$$

Spectrum of Vega[3]



[3]Hubble Space Telescope calibration data

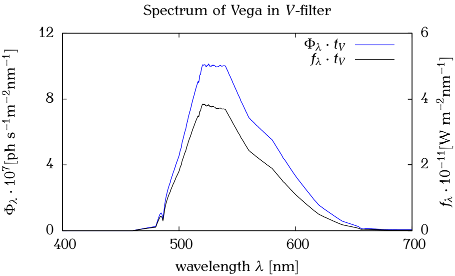
Crucial Relation of Photon Calibration

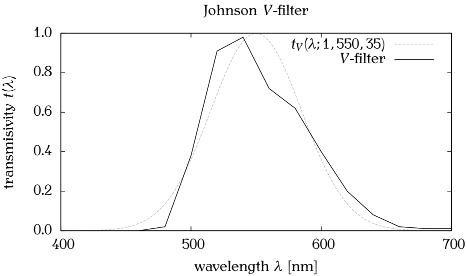
$$F = \int_0^\infty \Phi_\lambda(\lambda) \frac{hc}{\lambda}\, t(\lambda)\, \mathrm{d}\lambda.$$

How to get $\Phi_\lambda(\lambda)$ from F?

Johnson V-Filter

$$F = \int_0^\infty \Phi_\lambda(\lambda) \frac{hc}{\lambda}\, t(\lambda)\, \mathrm{d}\lambda = \int_0^\infty f_\lambda(\lambda)\, t(\lambda)\, \mathrm{d}\lambda.$$





$$t_V(\lambda) \approx t_0\, \mathrm{e}^{-(\lambda-\lambda_0)^2/2\delta^2}$$

Common values of V: $0 < t_0 < 1$, $\lambda_0 \approx 550\,\mathrm{nm}$, $\delta \approx 35\,\mathrm{nm}$

Gauss-Hermite Integration

General formula[4] (a_j are roots of Hermite polynomials $H_n(x)$):

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \sum_{j=1}^{n} H_j f(a_j) + E_n,$$

$$H_j = -\frac{2^{n-1} n! \sqrt{\pi}}{n^2 [H_{n-1}(a_j)]^2}, \quad E_n = \frac{n! \sqrt{\pi}}{2^n (2n)!} f^{(2n)}(\eta).$$

Application on Gaussian-like profiles:

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx = \sqrt{\pi}\, f(0) + E_1$$

$$E_1 = \frac{\sqrt{\pi}}{4} f''(\eta), \; \eta \in (-\infty, \infty).$$

[4] Ralston, Rabinowitz: A First Course in Numerical Analysis

Approximation of the Crucial Relation

$$F \approx \underbrace{\sqrt{2\pi}\, t(\lambda_0)\Phi_\lambda(\lambda_0)\frac{hc}{\lambda_0}\,\delta}_{\text{photon}} \approx \underbrace{\sqrt{2\pi}\, t(\lambda_0)\, f_\lambda(\lambda_0)\,\delta}_{\text{flux}}.$$

Relative errors of approximation of real filters are $\simeq 10^{-2}$.
Flux (also photon) magnitude relation:

$$\frac{F_V}{F_V^{\text{Vega}}} = \frac{\Phi_V}{\Phi_V^{\text{Vega}}} = 10^{-0.4\, m_V}.$$

Reference flux[5] for $m = 0$ (approximately Vega)

$$f_\lambda(\lambda = 550\,\text{nm}) = (3.56 \pm 0.01) \cdot 10^{-11}\,\text{W}\,\text{m}^{-2}\,\text{nm}^{-1}$$

gives

$$F_V^{\text{Vega}} = (3.1 \pm 0.01) \cdot 10^{-9}\,\text{W}\,\text{m}^{-2}$$

[5] Megessier (1995): A&A, **296**, 771

Common Astronomical Sources

magnitude V filter	flux $[\mathrm{W\ m^{-2}}]$	photon flux $[\mathrm{s^{-1} m^{-2}}]$	
-26	10^{3}	10^{20}	Sun
-13	10^{-4}	10^{15}	Full Moon
0	10^{-9}	10^{10}	Vega
5	10^{-11}	10^{8}	naked eye limit
10	10^{-13}	10^{6}	asteroids, comets
15	10^{-15}	10^{4}	quasars, blazars
20	10^{-17}	100	optical afterglows
25	10^{-19}	1	Earth telescope limit
30	10^{-21}	10^{-2}	invisibility limit

Everyday Applications

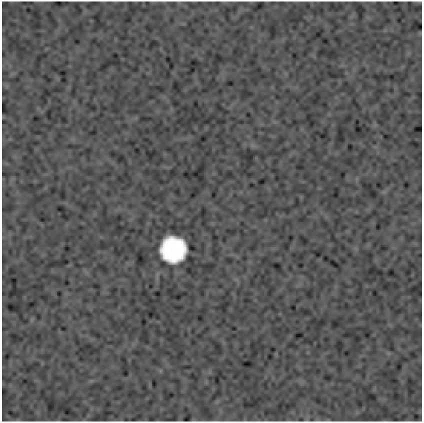
- Eye:
  diameter $\simeq 5\,\mathrm{mm}$ (area $2 \cdot 10^{-5}\,\mathrm{m}^2$), integration time $\simeq 0.1\,\mathrm{s}$:

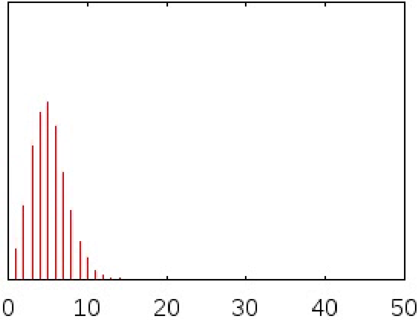
magnitude	photons	
0	10^4	Vega
5	10^2	naked eye limit

- Photo-voltaic panel:
  Measured by me: 2015-02-04, cca 12:30 (clear sky)
  Measurements: $I = 93\,\mathrm{mA}, 12.4 \times 6.4\,\mathrm{cm}$
  Estimate: $I = 12\,\mathrm{A\,m^{-2}} \approx 10^{20}\,\mathrm{e^-\,m^{-2}\,s^{-1}}$
  (no V-filter, spectral sensitivity unknown!)

Poisson's Nature of Photons

Photon Rain





Poisson Distribution

Let's, amount of photons is n, probability[6] observing of events λT is

$$P_n(\lambda T) = \frac{(\lambda T)^n \mathrm{e}^{-\lambda T}}{n!}, \quad (n = 0, 1 \dots)$$

- λ is event rate
- λT is number of occurred events per time period

Mean:

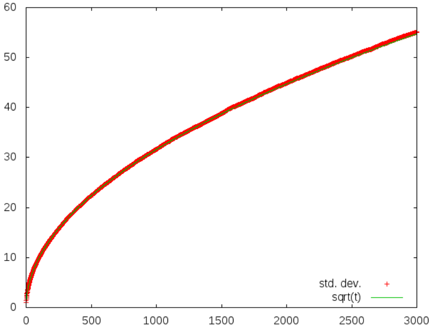
$$\bar{\lambda} = \lambda$$

Variance:

$$\sigma^2 = \lambda$$

Relative:

$$\frac{\sigma}{\bar{\lambda}} = \frac{1}{\sqrt{\lambda}}$$



[6] Rényi: Foundations of Probability (1970)

Mandel's Formula

$$P_n(T) = \mathrm{Tr}\left\{\hat{\varrho} : \frac{(\mu(T)\hat{a}^+\hat{a})^n}{n!} \exp(-\hat{a}^+\hat{a}\mu(T)) :\right\}$$

- $\mu(T) = \lambda T$ (open)
- $\mu(T) = (1 - \mathrm{e}^{-\lambda T})$ (closed)

Precise:

a) The value a has probability to by found $|\langle a|\psi\rangle|^2$

b) The state after measurement is $|a\rangle$

Imprecise:

a) $P(a) = \mathrm{Tr}\left\{\hat{A}(a)\hat{\varrho}_0\hat{A}^+(a)\right\}$

b) $\hat{\varrho}(a) = \hat{A}(a)\hat{\varrho}_0\hat{A}^+(a)/P(a)$

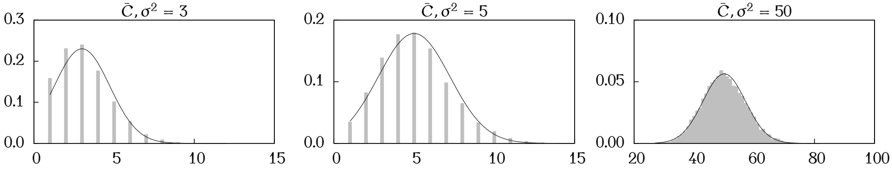
Law of Large Numbers

Asymptotic behaviour:

$$P_n(\lambda T) = \frac{(\lambda T)^n e^{-\lambda T}}{n!} \quad \xrightarrow{n \gg 1} \quad N(x, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\bar{x})^2}{2\sigma^2}\right)$$

with $C = \bar{\lambda} T$:

$$N(\bar{C}, \sqrt{\bar{C}}) = \frac{1}{\sqrt{2\pi\bar{C}}} \exp\left(-\frac{(C-\bar{C})^2}{2\bar{C}}\right)$$



Photon Calibration

Crucial Ratio of Calibration

Crucial approximation (again):

$$F \approx \sqrt{2\pi}\, t(\lambda_0)\Phi_\lambda(\lambda_0)\frac{hc}{\lambda_0}\,\delta$$

Count of expected photons in a filter V:

$$N_V = AT\,\Phi_V \approx \sqrt{2\pi}\,AT\,f_V\delta_V\,\frac{\lambda_V}{hc} \approx ATF_V\,\frac{\lambda_V}{hc}$$

- C is count of detected counts
- N is count of expected photons
- t effective transmissivity (efficiency)

$$\boxed{C = t\ N}$$

Maximum Likelihood

- probability distribution of a single point x_i is *a priory* $p(x_i|\theta)$
- like composing of probabilities $p = p_1 \cdot p_2 \dots p_N$, join distribution is $p(x_1, x_2 \dots x_N|\theta) = p(x_1|\theta) \cdot p(x_2|\theta) \dots p(x_N|\theta) \equiv L$
- parameter θ is determined for maximum of $p(x_1, x_2 \dots x_N|\theta)$

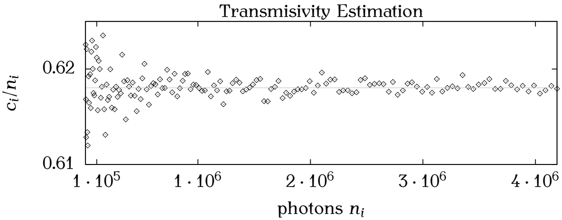
$$L = \prod_{i=1}^{N} p(x_i|\theta)$$

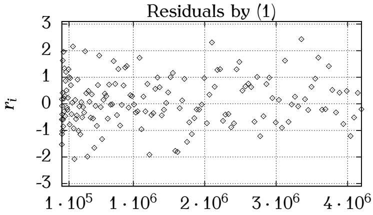
Common method to get maximum is use of derivation $\ln L$

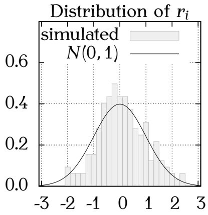
$$\frac{\partial \ln L}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=1}^{N} \ln p(x_i|\theta) = 0$$

Calibration of an Ideal Distribution

$$\frac{1}{\sqrt{2\pi\sigma_i^2(t)}} e^{-r_i^2/2}, \quad r_i = \frac{C_i - tN_i}{\sqrt{N_i + t^2 C_i}}, \quad \sigma_i = \sqrt{N_i + t^2 C_i} \qquad (1)$$





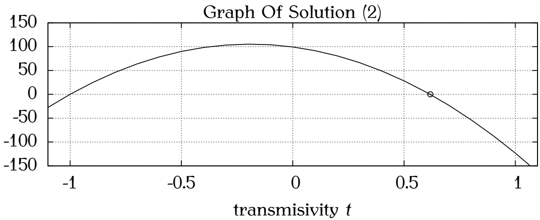


Solution of Calibration Equation

$$\ln L = -\sum_{i=1}^{M} \left[\frac{(C_i - tN_i)^2}{2\sigma_i^2(t)} + \ln \sqrt{2\pi\sigma_i^2(t)} \right].$$

with result

$$\sum_{i=1}^{M} \left[\left(\frac{C_i}{N_i} - t \right) (1 + t) - t \left(\frac{1}{N_i} + \frac{t^2}{C_i} \right) \right] = 0 \qquad (2)$$



Robust Statistics

Robust Statistical Methods

Robustness signifies insensitivity to small deviations from assumptions. – Peter J. Huber

Let[7] the observation x_i be independent, with common distribution F, and let $T_n = T_n(x_1, \dots x_n)$ be a sequence of estimates or test statistics with values in $\mathbb{R}^k$. This sequence is called robust at $F = F_0$ if the sequence of maps of distributions

$$F \to \mathcal{L}_F(T_n)$$

is equicontinous at F_0, that is, if for every $\varepsilon > 0$, there is a $\delta > 0$ and n_0 such that, for all F and all $n \geq n_0$,

$$d_*(F_0, F) \leq \delta \implies d_*(\mathcal{L}_{F_0}(T_n), \mathcal{L}_F(T_n)) \leq \varepsilon.$$

[7] Huber: Robust Statistics (2004)

Properties of Robust Methods

Kinds:

- M-estimates (maximum likelihood)
- R-estimates (rank)
- L-estimates (linear combination)

Properties:

- The definition is equivalent to the weak convergence of T_n,
- insensitive to outliers (by unexpected errors, apparatus defects, cosmics, ...),
- equivalent to least square for well noised data (the same dispersion),
- ideal for machine processing.

Robust Maximum Likelihood

Use of ML for a robust function:

$$L = \prod_{i=1}^{N} \frac{1}{s}\, f\left(\frac{C_i - tN_i}{s\sigma_i}\right)$$

substitutions

$$\psi(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln f(x), \quad r_i = \frac{C_i - tN_i}{\sqrt{C_i + t^2 N_i + \sigma_i^2 + \dots}}$$

Simultaneous estimation of t, σ:

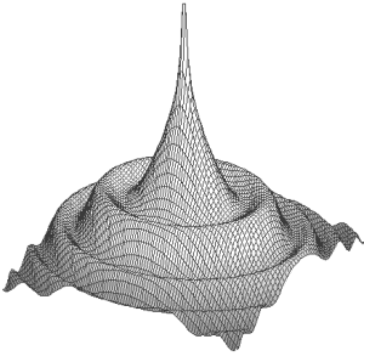
$$\sum_i \psi\left(\frac{r_i}{s}\right) = 0$$

$$\sum_i \left[\psi\left(\frac{r_i}{s}\right)\left(\frac{r_i}{s}\right) - 1\right] = 0$$

Advertising

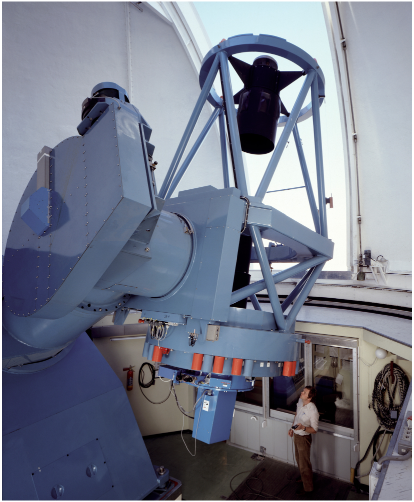
Munipack

- photometry corrections (bias, flat-field)
- astrometry (including matching)
- full photometry calibration (photon-based, colour system transformations, atmospheric corrections)
- robust statistical estimators
- Virtual observatory access
- basic FITS utilities
- command-line and GUI interface
- Open source (Fortran and C++), GPL



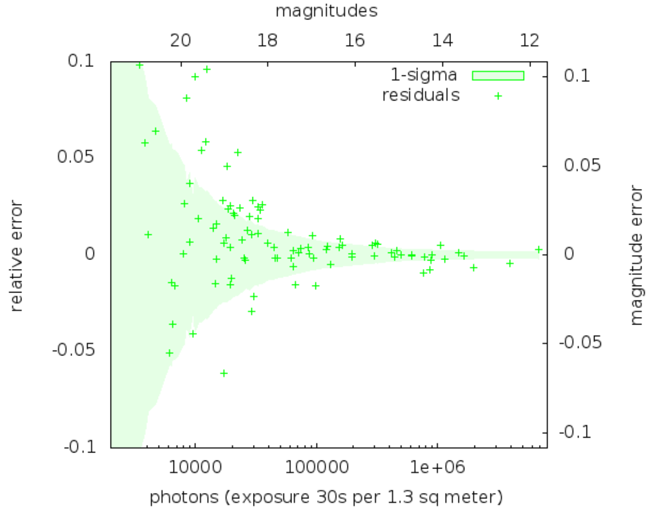
Applications

Danish 1.54 m Telescope

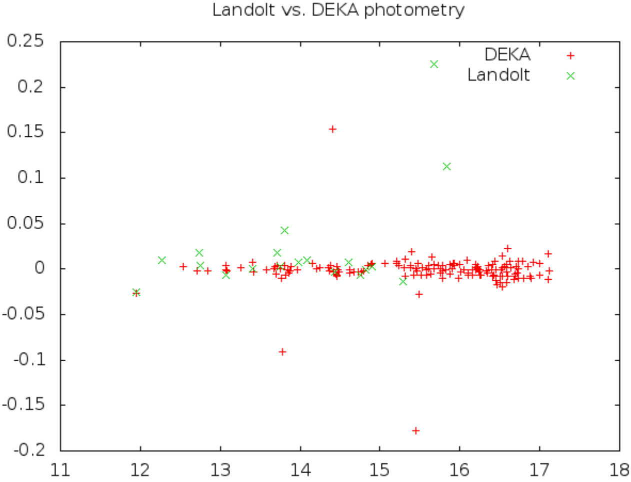


- Huge 1.54 m primary mirror
- La Silla, 2.4 km above sea
- Ritchey-Chrétien Reflector
- since 1979 (Niels Bohr Institute), 2012 (+ Astro-Institutions, CZ)

Residuals on Calibration Fields



DK154 Photometry Catalogue



Conclusions

Advantages of Photon Counting

Features:

- *Clear framework* on base of standard methods
- Applicable for any bandwidth, wavelength, instrument
- Easy application of robust methods
- Absolute calibration is natural part of one.
- Powerful handling of atmospheric effects

Advances:

- atmospheric reddening
- stellar profiles by atmospheric turbulence
- transformations of colour systems

And Now for Something Completely Different

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