Exam problems for the course
Statistical physics and thermodynamics
fall semester 2008.

These are hand in assignments for the course in "Statistical physics and thermodynamics" given at Masaryk University at the fall semester 2008. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 20 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 20 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. Consider a system consisting of two particles, each of which can be in any one of three quantum states of respective energies, 0, ϵ and 3ϵ. The system is in contact with a heat reservoir at temperature T. (**2p**)
   (a) Write an expression for the partition function Z if the particles obey classical MB statistics and are considered distinguishable.
   (b) What is Z if the particles obey BE statistics?
   (c) What is Z if the particles obey FD statistics?
2. For both the canonical and the grand canonical ensemble, prove that
$$S = -\sum_r P_r \ln P_r \tag{1}$$
   where P_r is the probability to find the system in microstate $|r>$. (**3p**)
3. Use the result of the previous exercise to prove that for an ideal quantum gas
$$S = -\sum_i \left(\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i) \ln(1 \pm \bar{n}_i)\right), \tag{2}$$
   where the upper sign is for bosons and the lower sign is for fermions and $\bar{n}_i$ is the *average* occupation number of 1-particle energy level i. (**6p**)
4. Consider an isolated system of N noninteracting particles in which the energy of each particle can assume two and only two distinct values, 0 and ϵ ($\epsilon > 0$). Denote by n_0 and n_1, the occupation numbers of the energy level 0 and ϵ respectively. The total energy of the system is E. (**3p**)
   (a) Find the entropy of such a system.
   (b) Find the most probable values of n_0 and n_1, and find the mean square fluctuations of these quantities. ($\Delta n^2 =< (n- < n >)^2 >$).

(c) Find the temperature as a function of E, and show that it can be negative.

(d) What happens when a system of negative temperature is allowed to exchange energy with a system of positive temperature?

5. Consider a classical system of noninteracting *diatomic molecules* enclosed in a box of volume V at temperature T. The Hamiltonian for a *single* molecule can be taken to be

$$H = \frac{1}{2m}\left(\mathbf{p}_1^2 + \mathbf{p}_2^2\right) + \frac{1}{2}K\left(\mathbf{r}_1 - \mathbf{r}_2\right)^2$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are the momenta and coordinates of the two atoms in the molecule. (**4p**)

(a) Find the free energy F of the system.

(b) Find the specific heat at constant volume.

(c) Find the mean square molecule diameter $< |\mathbf{r}_1 - \mathbf{r}_2|^2 >$

6. A box is separated into two compartments by a freely moving sliding wall. Two ideal Fermi gases are placed into the two compartments, numbered 1 and 2. The particles in compartment 1 have spin $\frac{1}{2}$, while those in compartment 2 have spin $\frac{3}{2}$. They all have the same mass. Find the equilibrium relative density of the two gases at $T = 0$ and $T \to \infty$. (**4p**)

7. Show that the entropy per photon in blackbody radiation is independent of the temperature, and that in d spatial dimensions can be written as

$$\frac{S}{N} = \frac{\zeta(d+1)}{\zeta(d)}.$$

Show also that the answer would have been $d + 1$ if the photons would have obeyed Boltzmann (classical) statistics. (**4p**)

8. Values for the specific heat of liquid He^4 are given in the table

T [K]	$C_V \left[\frac{J}{gK}\right]$
0.60	0.0051
0.65	0.0068
0.70	0.0098
0.75	0.0146
0.80	0.0222
0.85	0.0343
0.90	0.0510
0.95	0.0743
1.00	0.1042

(The values are obtained along the vapor pressure curve of liquid He^4 but you may assume that they are not too different from the values of C_V at the same temperature.)

(a) Show that the behavior of the specific heat at very low temperatures is characteristic of that of a phonon gas.

(b) Find the velocity of sound in liquid He^4 at low temperatures.

(**4p**)

9. We have seen that the thermodynamic behavior of solids is governed by waves in the lattice structure of the solid known as phonons. Similarly the two dimensional waves on the interface between a liquid and its vapor can be described by harmonic normal modes known as ripplons each having a singular direction of polarization perpendicular to the interface. The dispersion curve for these elementary excitations is isotropic and given by $\epsilon = \hbar\omega = bk^{\frac{3}{2}}$ where $k = \sqrt{k_x^2 + k_y^2}$. For a square sample with dimensions $L \times L$ the allowed wavevectors are $\bar{k} = \frac{2\pi}{L}(n_x, n_y)$ where n_x and n_y can take on all positive and negative integer values. Find an expression for the ripplon contribution to the constant area heat capacity. How does the heat capacity depend on T? Sketch the result. (**4p**)

10. When a diatomic molecule vibrates, its moment of inertia depends to a small extent on its vibrational state. Consequently. the rotational and vibrational motions are not completely independent. Under suitable conditions, the spectrum of vibrational and rotational energies can be approximated as

$$E_{n,l} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2I}l(l+1) + \alpha l(l+1)\left(n + \frac{1}{2}\right) \tag{3}$$

where the first two terms correspond to vibrational and rotational motion respectively, and the last term is a small correction that arises from the interdependence of vibrations and rotations. The various molecular constants satisfy

$$\hbar\omega >> \frac{\hbar^2}{2I} >> \alpha. \tag{4}$$

For an ideal gas of such molecules, compute the energy for temperatures in the range $\hbar\omega > T > \frac{\hbar^2}{2I}$. (**5p**)