Photonic crystals composed of Eaton lenses and invisible lenses

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We propose a photonic crystal composed of Eaton lenses and invisible lenses, respectively. We show that for certain frequency ranges the photonic crystal of Eaton lenses is transparent whereas the crystal composed of invisible lenses cannot be penetrated by the light waves. This is exactly the opposite behaviour than one can observe within geometrical optics where Eaton lens works as a perfect retroreflector while the invisible lens is transparent.

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I. INTRODUCTION

Eaton lens [1] and invisible lens [2–4] belong to the family of interesting optical devices, the so-called absolute optical instruments [5, 6] that are capable of creating a stigmatic (aberration free) image of a spatial region. Absolute instruments have attracted a lot of attention recently because of their suspected capability of providing super-resolution imaging not limited by diffraction [7] which, however, has not been confirmed [8].

In this paper, we propose photonic crystals (PC) composed of Eaton lenses and invisible lenses, respectively. We show that arranging the individual elements to form a PC can change their properties dramatically. In particular, a photonic crystal composed of Eaton lenses, each of which normally works like a perfect retroreflector sending light back to its source, can be partially transparent for light with specific frequency ranges. At the same time, a PC composed of invisible lenses which are perfectly transparent within geometrical optics, can be impassable for light with the same wavelength. This way, the PC composed of the tiny absolute instruments behave completely opposite than expected.

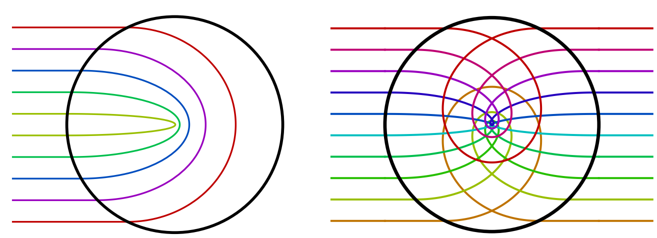
The paper is organised as follows. In Sec. II we recall some of the properties of Eaton lens and invisible lens and in Sec. III we introduce photonic crystals composed of them. In Sec. IV we analyse the photonic band structure of these PCs numerically. In Sec. V we present results of numerical simulations of waves in the PC and conclude in Sec. VI.

II. EATON LENS AND INVISIBLE LENS

Eaton lens [1] has a radially-symmetric refractive index profile given by the formula

$$n(r) = \sqrt{\frac{2a}{r} - 1} \tag{1}$$

for $r \leq a$, where r is the radial coordinate (either in 2D or 3D for the respective version of the lens) and a is the radius of the lens. This lens is closely related to Kepler problem in classical mechanics by the optical-mechanical analogy [9] and ray trajectories within the lens form confocal half-ellipses with their focus located at the lens centre, see Fig. 1(a). The refractive index diverges at the lens cent-er, $r \to 0$, but this singularity can be eliminated by the method of so-called transmutation of singularities [10]. Within the regime of geometrical optics, Eaton lens surrounded by vacuum behaves like an ideal retro-reflector, sending light rays back to their source, see Fig. 1(a). This has been demonstrated experimentally in a microwave regime on a transmuted 2D version of Eaton lens formed by a metamaterial [11].



(a) (b)

FIG. 1. (a) Rays in Eaton lens and (b) invisible lens.

Invisible lens (or, alternatively, invisible sphere) has a radially-symmetric refractive index profile given by the formula

$$n(r) = \left(Q - \frac{1}{3Q}\right)^2, \quad Q = \sqrt[3]{-\frac{a}{r} + \sqrt[2]{\frac{a^2}{r^2} + \frac{1}{27}}} \tag{2}$$

for $r \leq a$, with the meaning of r and a as before. This index profile was proposed independently in refs. [3] and [2]. Light rays in the invisible lens make a loop around its centre, leaving in the original direction, see Fig. 1(b). This makes the lens effectively invisible within geometrical optics, although it does not have potential to hide another object and work as an invisible cloak. For waves, the lens

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introduces a constant phase delay to the rays that have passed through it, disturbing the light wavefronts. For a discrete set of resonant frequencies, however, this delay equals an integer multiple of 2π and the lens is practically invisible also within wave optics [4].

III. PHOTONIC CRYSTALS

Having recalled Eaton lens and invisible lens, we can proceed to proposing a photonic crystal composed of them. Imagine that we densely arrange identical two-dimensional Eaton lenses into a rectangular grid embedded in air with refractive index $n \approx 1$, see Fig. 2. We will call the resulting photonic crystal an "Eaton-lens PC". Similarly we define an "invisible-lens PC" by replacing Eaton lenses by invisible lenses.

Suppose we illuminate the Eaton-lens PC normally, as in Fig. 2. In the situation when geometrical optics applies, all rays will make elliptic loops around the Eaton lens centres and will return back to the left, so the light will not penetrate even the first layer of Eaton lenses just as seen in Fig. 2. On the other hand, in the same regime, the invisible-lens PC will behave like a transparent structure because each light ray will make a loop in each invisible lens that it penetrates and leave it in the original direction, see Fig. 3.

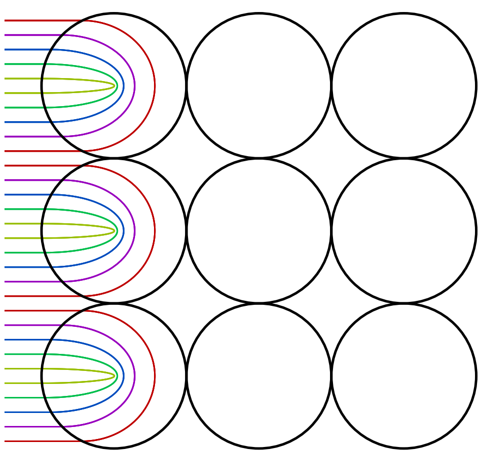

FIG. 2. Section of a photonic crystal composed of Eaton lenses illuminated from the left. Within the validity regime of geometrical optics, all light will be reflected.

However, in case that the wavelength λ is comparable with the Eaton lens of radius a, the behaviour of these PCs can be very different than in the geometrical optics regime. We show this by investigating the photonic band structure of the PCs.

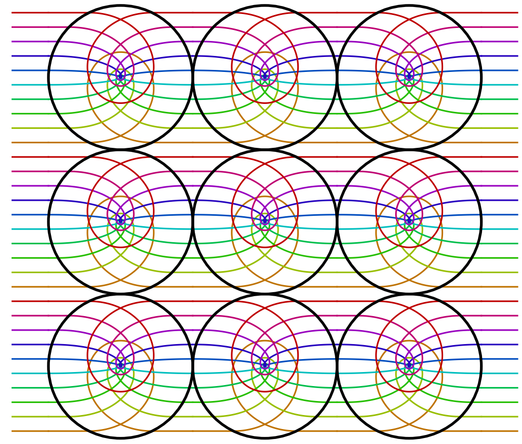

FIG. 3. Section of a photonic crystal composed of invisible lenses illuminated from the left. Within the validity regime of geometrical optics, all light will be transmitted.

IV. PHOTONIC BAND STRUCTURE OF THE PCS

We calculate the band structure of the photonic crystals defined in the previous section by the standard tools [12]. For simplicity, we will consider just the two-dimensional situation and scalar waves described by the function $u(\vec{r})$. The wave obeys the Helmholtz equation

$$\Delta u + k_0^2 n^2 u = 0\,, \tag{3}$$

where $k_0 = \omega/c$. The elementary cell of the PC (we denote is S) is a square of the side length $2a$; we orientate the axes x, y of a Cartesian coordinate system along the edges of this square, and place its origin at the centre of one of the lenses. Using Bloch's theorem, we write

$$u(\vec{r}) = \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}} v(\vec{r})\,, \tag{4}$$

where v is a function periodic in both x and y with period $2a$. We then get the following equation for v:

$$\Delta v + 2\mathrm{i}\vec{k}\nabla v - k^2 v + k_0^2 n^2 v = 0\,. \tag{5}$$

Next we expand v into the Fourier series

$$v = \sum_{p,q\in\mathbb{Z}} c_{pq} \mathrm{e}^{\mathrm{i}b(px+qy)}\,, \tag{6}$$

where $b = \pi/a$ is the reciprocal grating parameter. We substitute Eq. (6) into Eq. (5), multiply by $\mathrm{e}^{-\mathrm{i}b(p'x+q'y)}$ and integrate over the elementary cell S, i.e., from $-a$ to a in both x and y directions. This way we get the following relations for the coefficients c_{pq}:

$$[(k_x + bp)^2 + (k_y + bq)^2] c_{pq} = k_0^2 \sum_{p',q'\in\mathbb{Z}} c_{p'q'} N_{p-p',q-q'}\,, \tag{7}$$

where we have denoted

$$N_{p,q} = \frac{1}{4a^2}\int_S n^2(x,y)\mathrm{e}^{-\mathrm{i}b(px+qy)}\,\mathrm{d}x\,\mathrm{d}y \tag{8}$$

To use this result practically, we truncate the series in Eq. (6) by a certain index $p_{\max}$ such that the indexes p, q go from $-p_{\max}$ to $p_{\max}$. This gives a finite number $(2p_{\max}+1)^2$ of equations (7) that lead to the eigenvalue problem for the unknown $k_0 = \omega/c$ as a function of $\vec{k}$. Its solution gives the band structure of the photonic crystals.

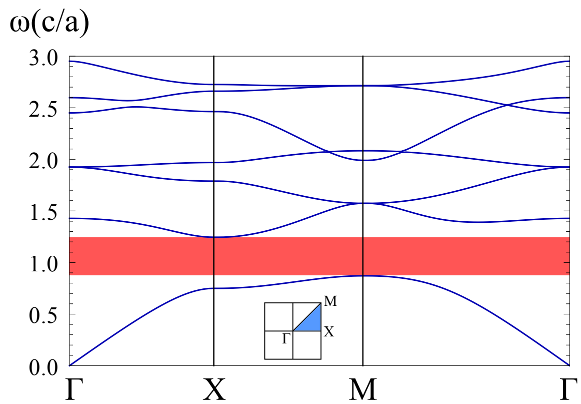

FIG. 4. Band structure of the Eaton-lens PC calculated with $p_{\max} = 15$. The inset shows the first Brillouin zone and the wavevector $\vec{k}$ runs around the blue triangle. The bandgap is marked in red.

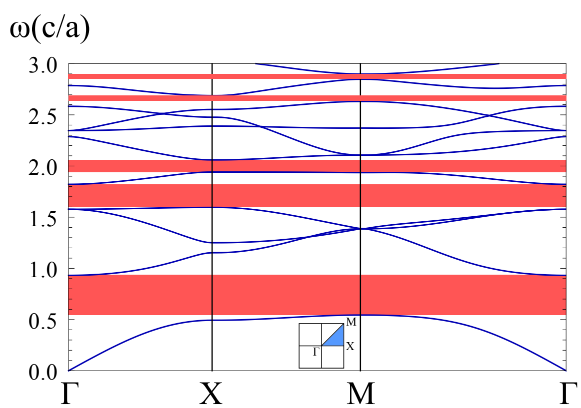

FIG. 5. Band structure of the invisible-lens PC calculated with $p_{\max} = 15$; other notation is the same as in Fig. 4.

The band structure calculated numerically for $p_{\max} = 15$ is presented in Figs. 4 for Eaton-lens PC and in Fig. 5 for the invisible-lens PC. Clearly, there are bandgaps present in the spectrum, showing that for certain ranges of frequencies the electromagnetic waves cannot propagate in the PC but can at most penetrate there evanescently. This may be surprising for the invisible-lens PC because it is transparent within the geometrical optics regime as discussed in the previous section.

To investigate this phenomenon in more detail, we focus on the situation when a wave incides normally on the PC from the left (which, in the geometrical optics regime, would correspond to the situation in Figs. 2 and 3). Then the wave in the crystal must be periodic in the vertical (y) direction with period $2a$, which means that k_y, the y-component of the wavevector $\vec{k}$ from Eq. (4), must be an integer multiple of b. However, as all wavevectors $\vec{k}$ can be reduced to the first Brillouin zone, we can set $k_y = 0$ without loss of generality. Therefore to see in which frequency ranges the wave normally incident on the PC cannot propagate within it, we need to check the band structure on the x-axis in the k-space, which corresponds to the segment Γ-X in Figs. 4 and 5. For the bandgaps corresponding to these segments we can expect that the PC will be completely reflective. We will verify this in the following section.

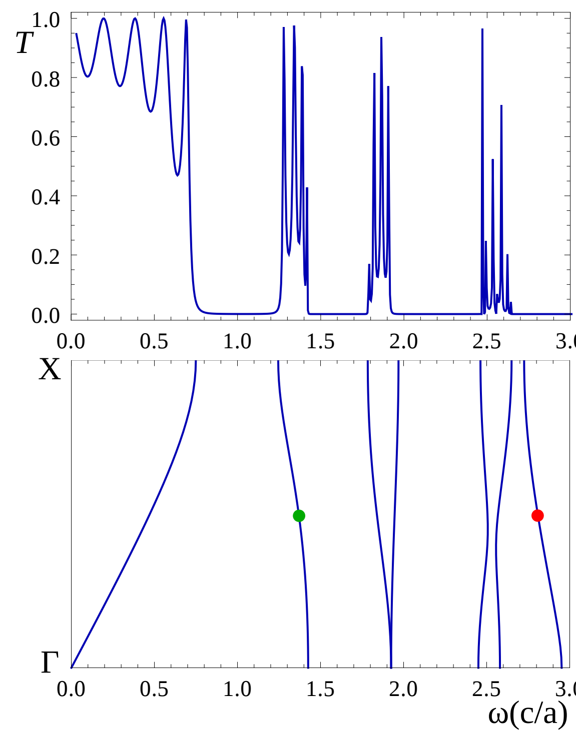

FIG. 6. Transmissivity of a 5-layer invisible-lens PC calculated using COMSOL with the band structure in the Γ–X direction for comparison. We see that the bandgaps correspond to regions with a very low transmissivity, as expected. The red and green dot mark the seventh and second band, respectively, and correspond to the modes shown in Fig. 8.

V. TRANSMISSIVITY OF THE PHOTONIC CRYSTALS

To verify the results from the previous section by an independent method, we have calculated the transmissivity of the photonic crystals using the finite-element software

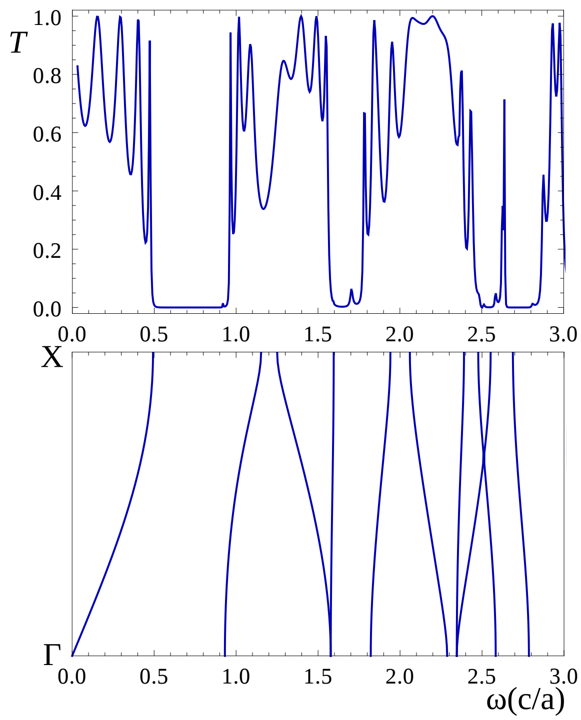

FIG. 7. The same as Fig. 6, but for the invisible-lens PC.

Comsol. We simulated the situation of a normal incidence of a plane wave on the PC composed of five layers of Eaton or invisible lenses, respectively. The results are shown in Figs. 6 and 7. For comparison, the 90°-rotated band structure in the Γ–X direction is shown under the graphs.

First, Figs. 6 and 7 reveal that the near-zero transmissivity regions indeed correspond very well to the bandgaps in the band structure. However, there are also intervals of frequencies (e.g., $\omega a/c \in [2.72, 2.95]$ for the Eaton-lens PC) where the transmissivity is close to zero but still there is a clear band (e.g. the seventh band in Fig. 6 marked by a red dot). It is not difficult to find an explanation for this phenomenon: the corresponding modes are antisymmetric with respect to the line $y = 0$ (see Fig. 8) and therefore they cannot be excited by a wave normally incident on the PC that is symmetric with respect to $y = 0$.

Second and more importantly, from Figs. 6 and 7 we observe that there are certain spectral regions (e.g. $\omega a/c \in [0.5, 0.75]$) where the Eaton-lens PC has a large transmissivity while the transmissivity of the invisible-lens PC is negligible. This way, the behaviour of the the PCs in these spectral regions is exactly opposite compared to the geometrical optics regime, and the two PCs completely interchange their behaviour. This result may be quite surprising and it shows that behaviour of certain optical structures can be strikingly different for waves and rays, the difference coming from the wave interference.

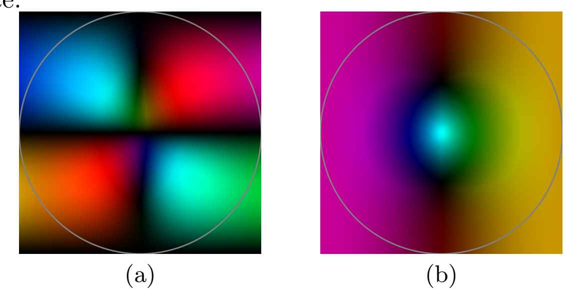

FIG. 8. The numerical solutions of Eq. (4) for the Eaton-lens PC with $\vec{k} = (0.25\,\pi/a, 0)$ for (a) the seventh band and for (b) the second band for comparison that correspond to the red and green dot in Fig. 6, respectively. The wave amplitude and phase is encoded into brightness and hue, respectively. The seventh band modes cannot be excited by the wave incident on the PC along the x-axis due to their antisymmetry, therefore there is an almost zero transmissivity in the corresponding frequency range.

VI. CONCLUSION

We have analysed some features of the band structure of photonic crystals composed of tiny absolute optical instruments (Eaton lenses and invisible lenses), and have shown that these PCs sometimes behave in a completely opposite way in the geometrical-optics and wave-optics regime. In particular, in certain frequency ranges a PC composed of Eaton lenses is transparent for waves while it is completely reflective in the geometrical optics regime, and a PC composed of invisible lenses is completely reflective for waves while it is completely transparent in the geometrical optics regime.

One possible realisation of the PCs discussed in this paper could be based on the idea of geodesic lenses: the Eaton or invisible lenses could be replaced by thin 2D waveguides shaped into a special profile [13, 14]. This would reduce the required values of refractive index to a finite range, which would make building these devices much easier.

VII. ACKNOWLEDGEMENTS

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