Exam problems for the course
Introduction to General Relativity
fall semester 2021.

These are hand in assignments for the course in "Introduction to General Relativity" given at Masaryk University at the fall semester 2021. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 40 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 40 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. In this problem I want you to show the existence of the Coriolis and Centrifugal forces using techniques from General Relativity. Start with a spherical coordinate system

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

and the geodesic equation

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = 0 \tag{2}$$

where the indices take values in $\{r, \theta, \phi\}$. Shift the connection terms to the right hand side to interpret them as force terms. To go to a rotating system, redefine the coordinate $\phi = \Omega t + \varphi$ and then collect terms quadratic in Ω and linear in Ω. Show that

$$\begin{aligned} \hat{e}_z \times \partial_r &= \frac{1}{r}\partial_\phi && (3)\\ \hat{e}_z \times \partial_\theta &= \cot\theta\partial_\phi && (4)\\ \hat{e}_z \times \partial_\phi &= -r\sin^2\theta\partial_r - \sin\theta\cos\theta\partial_\theta && (5) \end{aligned}$$

and use this together with the definitions

$$\begin{aligned} \bar{r} &= r\partial_r && (6)\\ \bar{v} &= \dot{r}\partial_r + \dot{\theta}\partial_\theta + \dot{\varphi}\partial_\phi && (7) \end{aligned}$$

to rewrite the equations in vector form and identify the Coriolis and Centrifugal force terms. (**10p**)

2. To study what happens inside a black hole, introduce a new time coordinate

$$\bar{t} = t + 2M\ln\left|\frac{r}{2M} - 1\right| \tag{8}$$

How does the metric in the new coordinate system look like? Show that the metric will have full rank even when $r = 2M$ so that there is no coordinate singularity.

Find the geodesic equation for a radial geodesic. Show that you can combine the two equations into the equation

$$\left(\frac{2M}{r} - 1\right)\ddot{t} + \frac{2M}{r}\ddot{r} - \frac{2M}{r^2}\dot{t}\dot{r} - \frac{2M}{r^2}\dot{r}^2 = 0 \tag{9}$$

which can be easily integrated once (it is a total derivative).

Use the solution to get rid of the $\dot{t}$ dependence in the equation for r and integrate also this equation by finding an appropriate integrating factor.

The r equation can be used to calculate how long someone starting at the horizon $r = 2M$ with zero velocity would live before reaching the singularity. Calculate this time for a solar mass black hole and for a black hole with $7 \cdot 10^9$ solar masses like the black hole at the center of the galaxy M87. (**10p**)

3. What is the difference in proper time measured by an astronaut spending a year at the ISS and you staying at the surface of the Earth? Who measures the longest time? Why is this result strange? Can you explain how this can be? (**10p**)

4. Derive the geodesic equation for motion around the center in the Schwarzschild space time in the plane $\theta = \frac{\pi}{2}$. Show that they can be integrated to

$$\begin{aligned} \dot{t} &= \frac{E}{1 - \frac{2M}{r}} && (10) \\ \dot{\phi} &= \frac{L}{r^2} && (11) \\ \dot{r}^2 &= E^2 - V(r) && (12) \end{aligned}$$

where E and L are integration constants. Find an explicit expression for $V(r)$ both in the massive and the massless case. The radial equation looks like the energy conservation equation for a particle in a one dimensional effective potential. If we want the orbit to be circular we need $\dot{r} = 0$ so r should be in a minimum of the potential. Analyze the different possibilities as we change the constant L (the angular momentum) both in the massive and massless case. (**10p**)

5. Assuming a static metric with spherical symmetry

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right), \tag{13}$$

solve Einstein's equations with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} \tag{14}$$

to show that

$$f(r) = h^{-1}(r) = 1 - \frac{2M}{r} + \frac{\Lambda}{3}r^2 \tag{15}$$

(**10p**)

6. Consider the 4-dimensional space-time described by the Kasner metric:

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2 \tag{16}$$

Show that for this to be a solution to the Einstein vacuum equations we need $p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1$. Are there any singularities. How does the geometry of a spatial slice (t constant) behave as a function of time? (**10p**)

7. In the movie "2001: A space oddyssey" they create an artificial gravitational field by having the spaceship rotate. This can be described by the (three dimensional) metric

$$ds^2 = -(1 - \omega^2\rho^2)dt^2 + d\rho^2 + \rho^2 d\theta^2 + 2\omega\rho^2 dt d\theta, \tag{17}$$

where ω is a constant. An observer in the spaceship will think of the direction in increasing ρ as "down" and θ as "left" and "right". (The directions "backwards" and "forwards" have been left out, there is no room for them on the spaceship.) If an observer drops something from the position (ρ_0, θ_0), find the trajectory of the object in the (ρ, θ) plane. Comment on the result. How can we detect that we are not on earth? (**10p**)

8. A Killing vector field is a vector field satisfying the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{18}$$

Show that

$$\nabla_\mu \nabla_\nu \xi_\sigma = R^\alpha_{\mu\nu\sigma}\xi_\alpha \tag{19}$$

(remember that $R^\alpha_{\mu\nu\sigma} + R^\alpha_{\nu\sigma\mu} + R^\alpha_{\sigma\mu\nu} = 0$). Killing vectors allow us to define conserved quantities in General Relativity. Assume that a geodesic observer has 4-velocity u^μ. Show that $\xi_\mu u^\mu$ is constant along his path. Show also that $j_\mu = T_{\mu\nu}\xi^\nu$ is a conserved current ($\nabla^\mu j_\mu = 0$) for any conserved energy momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$). Show that $\xi = \frac{\partial}{\partial t}$ is a Killing vector for the Schwarzschild space-time. Calculate explicitly $\xi_\mu u^\mu$ where u is the 4-velocity of a radially moving geodesic observer. What is the conserved quantity? (**10p**)