Exam problems for the course
Statistical physics and thermodynamics
fall semester 2010.

These are hand in assignments for the course in "Statistical physics and thermodynamics" given at Masaryk University at the fall semester 2010. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 24 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 24 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. For both the canonical and the grand canonical ensemble, prove that
$$S = -\sum_r P_r \ln P_r \tag{1}$$
where P_r is the probability to find the system in microstate $|r>$. (**3p**)

2. Use the result of the previous exercise to prove that for an ideal quantum gas
$$S = -\sum_i \left(\bar{n}_i \ln \bar{n}_i \mp (1 \pm \bar{n}_i)\ln(1 \pm \bar{n}_i)\right), \tag{2}$$
where the upper sign is for bosons and the lower sign is for fermions and $\bar{n}_i$ is the *average* occupation number of 1-particle energy level i. (**6p**)

3. For a classical ideal gas (Maxwell distribution), what is the probability that two randomly picked particles have a joint energy in the range $(E, E+dE)$? What is the mean value of the energy of two randomly picked particles? (**4p**)

4. A box is separated into two compartments by a freely moving sliding wall. Two ideal Fermi gases are placed into the two compartments, numbered 1 and 2. The particles in compartment 1 have spin $\frac{1}{2}$, while those in compartment 2 have spin $\frac{3}{2}$. They all have the same mass. Find the equilibrium relative density of the two gases at $T = 0$ and $T \to \infty$. (**4p**)

5. Consider an ideal Bose gas confined in a box of height L which is in a uniform gravitational field (of acceleration g). Show that the phenomenon of Bose-Einstein condensation sets in at a temperature T_c given by
$$T_c \approx T_c^{(0)}\left[1 + \frac{8}{9}\frac{1}{\zeta(\frac{3}{2})}\left(\frac{\pi mgL}{T_c^{(0)}}\right)^{\frac{1}{2}}\right] \tag{3}$$
where $T_c^{(0)}$ is the critical temperature in the absence of a gravitational field and we have assumed that $mgL \ll T_c^{(0)}$. It is useful to know that for

small negative y we may write

$$B_{\frac{1}{2}}(y) \approx \Gamma(\frac{3}{2})\Big(\zeta(\frac{3}{2}) - 2\Gamma(\frac{1}{2})\sqrt{-y} + \dots\Big) \tag{4}$$

$$\Gamma(\frac{1}{2}) = \sqrt{(\pi)} \tag{5}$$

(**6p**)

6. When a diatomic molecule vibrates, its moment of inertia depends to a small extent on its vibrational state. Consequently. the rotational and vibrational motions are not completely independent. Under suitable conditions, the spectrum of vibrational and rotational energies can be approximated as

$$E_{n,l} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2I}l(l+1) + \alpha l(l+1)\left(n + \frac{1}{2}\right) \tag{6}$$

where the first two terms correspond to vibrational and rotational motion respectively, and the last term is a small correction that arises from the interdependence of vibrations and rotations. The various molecular constants satisfy

$$\hbar\omega >> \frac{\hbar^2}{2I} >> \alpha. \tag{7}$$

For an ideal gas of such molecules, compute the energy for temperatures in the range $\hbar\omega > T > \frac{\hbar^2}{2I}$. (**5p**)

7. Show that the equation of state for a nonrelativistic ideal Boson gas in the nondegenerate limit can be written as

$$\frac{PV}{NT} = \Big(1 + B(T)\frac{N}{V} + \mathcal{O}(\frac{N^2}{V^2})\Big) \tag{8}$$

and calculate $B(T)$ explicitly. Compare to the $B(T)$ one gets from the nonideal classical gas in the high temperature limit and where the interaction potential between two particles is

$$V = \begin{cases} \infty & r < \sigma \\ -u_0 & \sigma < r < r_0 \\ 0 & r_0 < r \end{cases} \tag{9}$$

You may use without proof that

$$B_n(y) = \Gamma(n+1)\Big(e^y + \frac{1}{2^{n+1}}e^{2y} + \dots\Big) \tag{10}$$

(**6p**)

8. Consider an ideal Fermi gas, with energy dispersion relation $\epsilon \propto p^s$, contained in a box of volume V in a space of n dimensions. Show that, for this system

$$PV = \frac{s}{n}E \tag{11}$$

and that the equation for an adiabatic (S and N constant) process is $PV^{1+s/n} = \text{const}$. Show also that in the $T \to \infty$ limit $C_V = \frac{n}{s}N$. (**5p**)

9. The velocity of sound in a fluid is given by the formula

$$v = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \tag{12}$$

where ρ is the mass density of the fluid. Show that for a nonrelativistic ideal Bose gas

$$v^2 = \frac{5}{9}\left\langle u^2 \right\rangle = \frac{10T}{9m}\frac{B_{3/2}}{B_{1/2}} \tag{13}$$

where $\left\langle u^2 \right\rangle$ is the mean square speed of the particles in the gas. (**5p**)

10. Assume that there exist a system where the maximum number of particles in each state is a positive integer l. Find an expression for the mean occupation number of a quantum ideal gas of such a system. (**4p**)