# Hand-in assignments in QED, fall semester 2003.

These are hand in assignments for the course in Quantum electrodynamics at the Masaryk University in the fall of year 2003. They are the first part of the requirement of the course, the second being an oral exam. Each problem can give points as indicated and for a grade of 1 you need more than 85% of the points, for a grade of 2 a total of 65%-85% is required and for a grade of 3 a total of 50%-65% is required. The problems should be handed in minimum one week before the oral exam. **Do not leave out any part of the calculations and motivate your assumptions and approximations carefully**. You may answer in Czech *or* English.

1. If $u(p)$ satisfies the momentum space Dirac equation $(\not{p} - m)u(p) = 0$, what is the equation satisfied by $\bar{u}(p)$? Use this to show that

$$\bar{u}^{(\alpha)}(p)\not{q}u^{(\beta)}(p) = \frac{p \cdot q}{m}\delta^{\alpha\beta}.$$

   (**4p**)

2. A more realistic model of the nucleus would be to treat it as a spherical charge distribution $Ze\rho(r)$, where

$$\int d^3x\rho(r) = 1.$$

   Show that the differential cross section from scattering in the electric field from such a nucleus is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_M |\tilde{\rho}(\mathbf{q})|^2 ,$$

   where $\left(\frac{d\sigma}{d\Omega}\right)_M$ is the cross section for scattering from a point like source (called the Mott cross section). $Ze\tilde{\rho}(\mathbf{q})$ is the Fourier transform of the charge distribution and $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ is the difference between incoming and outgoing momenta. $\tilde{\rho}$ is called the form factor of the charge distribution. (**4p**)

3. The $\omega$-meson (whose rest energy is 783 MeV) is an electrically neutral spin 1 particle that decays via strong interaction into $\pi^+ + \pi^- + \pi^0$ most of the time. Since a virtual process

$$\omega \rightarrow \text{pair of strongly interacting particles} \rightarrow \gamma \rightarrow e^+ + e^-,$$

is allowed, the $\omega$-meson is expected to decay into $e^+ + e^-$ some of the time. This decay interaction may be represented phenomenologically by

$$\mathcal{H}_{\text{int}} = ig\phi_\mu \bar{\psi}\gamma^\mu\psi,$$

where $\phi_\mu$ is the neutral vector field corresponding to the $\omega$ meson (vector since $\omega$ has spin 1). Experiments have indicated that the decay rate

$$\frac{\Gamma(\omega \to e^+ + e^-)}{\Gamma_{\text{tot}}} \approx 10^{-4},$$

where $\Gamma_{\text{tot}}$ is known to be about 10 MeV. Estimate $\frac{g^2}{4\pi}$. Express your final answer in the form

$$\frac{g^2}{4\pi} = (\text{dimensionless number}) \times \left(\frac{1}{137}\right)^2.$$

Hint: remember that for a massless vector particle (i.e. the photon), the polarization vector $\epsilon_\mu$ is always orthogonal to the four momentum and the time component $\epsilon_0 = 0$. In the massive case (such as in this problem) one looses the property that the time component of the polarization vector is always zero, but one still has the condition that it is orthogonal to the four momentum. In particular this means that in the rest system of the $\omega$, the polarization vector is an arbitrary three vector $\epsilon_\mu = (0, \epsilon)$.**(6p)**

4. Calculate the lifetime of positronium (the bound state between an electron and a positron) in the ground state. The ground state can exist both with total spin 0 and total spin 1 depending on how the spins of the electron and positron combine (and there is no orbital angular momentum in the ground state). The state with total spin 0 always annihilates into two photons and the state with total spin 1 always annihilates into three photons. Compute the lifetime of the state with total spin 0. Reason as follows: positronium can be thought of as a hydrogen atom but with the nucleus (the proton) replaced by a positron. In the ground state the electron and positron can be thought of as being almost at rest so the probability of annihilation can be calculated in the limit where the velocities go to zero.

- Calculate the total probability per unit time that the electron and positron annihilate when they meet in the limit that they have zero velocity. Remember that the incoming state is a total spin zero state and be careful not to over count the outgoing states (the first photon having momentum $\mathbf{k}$ should be counted as the *same* situation as if the second photon has momentum $\mathbf{k}$). Take the zero velocity limit as soon as possible to get simpler formulas and use that you know that the final result is spherically symmetric so that you can choose a nice coordinate system.
- What about if the initial state had been a total spin 1? Can you say anything about the annihilation probability?
- The probability you just calculated is the probability that the electron and positron annihilate when there is one of each of them in the whole universe (remember our normalization). We should rather calculate the probability with an electron density (that is electrons per *unit* volume) $\rho$. This means $\rho V$ electrons in the whole universe so the total probability is $w\rho V$ where $w$ is the number you just calculated.
- The electron density $\rho$ can be evaluated by using that the electron positron wave function is the same as the ground state of the hydrogen atom (with a modified Bohr radius of course).
- Now you have the total probability for positronium annihilation. The lifetime is defined as the average time of life you would measure if you started with a large set of positronium "atoms". It is given by the inverse of the probability. Can you think of an argument why?

Compare your result to the experimental value 125 $ps$. (**6p**)

5. Determine the total cross section at ultra relativistic energies for the process where an electron-positron pair is annihilated and converted into a muon and an anti-muon ($e^+ + e^- \to \mu^+ + \mu^-$). Assume that the incoming electrons are unpolarized and that we do not observe the spin of the outgoing muons. (**4p**)

6. Write down all diagrams contributing to Compton scattering at the next to lowest order ($e^4$). There should be 18 of them. Write down the

expression for the graphs involving a fermion running around in a "triangle" loop and show explicitly that the sum of them cancel (without doing any integrals). (**4p**)