Hand-in assignments in Advanced Quantum Mechanics, spring semester 2022.

These are hand in assignments for the course in Advanced Quantum mechanics at the Masaryk University in the spring of year 2022. They are the first part of the requirement of the course, the second being an oral exam. The problems should be handed in minimum one week before the oral exam. **Do not leave out any part of the calculations and motivate your assumptions and approximations carefully**. You my answer in Czech *or* English. The required minimum number of points is **25** evenly distributed over the different topics.

The formalism

1. Imagine that you are given a nonorthonormal basis $|i\rangle$ so that $B_{ik} = \langle i|k\rangle$ is a general invertible matrix. Use B to write an expression for the unity operator in this basis. Write an arbitrary state $|\psi\rangle$ in this basis, *i.e.* find an expression for c_k in

$$|\psi\rangle = \sum_k |k\rangle \, c_k$$

Also, if A is any operator, find an expression for the representation of this operator in the given basis

$$A = \sum_{k,l} |k\rangle \, A_{kl} \, \langle l|$$

(4p)

Propagators and Path Integrals

1. A charged particle moving in a one dimensional space with an electric field F has a Hamiltonian given by

$$\hat{H} = \frac{\hat{p}^2}{2m} - F\hat{x} \; .$$

Derive the propagator for this system in the following way: Since the Hamiltonian is independent of time, argue that you may write the time evolution operator as

$$\hat{U}(t, t') = e^{-\frac{i}{\hbar}(t-t')\hat{H}}$$

Use the Baker-Campbell-Hausdorff formula to write the time evolution operator as

$$\hat{U}(t,t') = e^{f(\hat{x})}e^{g(\hat{p})}$$

You may now calculate the configuration space propagator $\langle x|\,\hat{U}(t,t')\,|x'\rangle$ by cleverly inserting the unit operator. Evaluate the resulting integral to find the final expression. **(6p)**

2. Find the time evolution of the 1 dimensional harmonic oscillator state

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}},$$

using the harmonic oscillator propagator

$$\begin{aligned} K(x',t';x,t) &= \sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega\Delta t}} \times \\ &\exp\left\{\left(\frac{im\omega}{2\hbar\sin\omega\Delta t}\right)\left((x'^2+x^2)\cos\omega\Delta t - 2x'x\right)\right\} \end{aligned}$$

Interpret your result! (**4p**)

Relativistic quantum mechanics

1. If we interpret $\rho = \psi^\dagger\psi$ as the particle density, use the Dirac Hamiltonian to define a current $\mathbf{j}$ such that the equation of continuity is fulfilled

$$\partial_t\rho + \nabla\cdot\mathbf{j} = 0$$

In order to do this it is useful to have the relation

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

which you should prove. Calculate ρ and $\mathbf{j}$ for the plane wave $\psi(\mathbf{x}) = u(\mathbf{p})e^{-\frac{i}{\hbar}Et}e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$ normalized so that $\psi^\dagger\gamma^0\psi = 1$. Comment on your result. **(4p)**

2. Prove the following trace identities valid for any representation

$$\begin{aligned} \mathrm{tr}(\gamma^\mu) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \end{aligned}$$

using *only* the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and the properties of the trace. To prove the first and the third relation, it is useful to define the matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^2$$

and show that $\gamma_5\gamma_5 = 1$ and that it anticommutes with all the γ^μ. **(4p)**

3. In a two-dimensional space-time with coordinates (t, x) the Klein-Gordon equation looks like

$$\frac{1}{c^2}\partial_t^2\phi - \partial_x^2\phi + \left(\frac{mc}{\hbar}\right)^2 \phi = 0$$

derive the Dirac equation by using the fact that $(E - pc)(E + pc) = E^2 - p^2c^2$. What do the gamma matrices look like? Show that they satisfy the defining relations of a Clifford algebra. **(4p)**

4. Define the spin operator

$$\Sigma_k = \frac{1}{2i}\epsilon_{klm}\gamma^l\gamma^m$$

and find the eigenvectors and eigenvalues of Σ_3. Show that it does not commute with the Dirac Hamiltonian in general so it is not a conserved quantity. However, if we measure the spin along the direction of motion we may define the helicity operator

$$\frac{\mathbf{p} \cdot \mathbf{\Sigma}}{|\mathbf{p}|}$$

Show that it commutes with the Dirac Hamiltonian and thus is conserved. **(4p)**

Scattering theory

1. Imagine studying particle scattering in a two dimensional world (there are plenty of examples of effectively two dimensional physical systems in Condensed Matter Physics). How would one define the cross-section? What dimension (unit) would it have? (**2p**)

2. Show how the wavefunction in the 2D scattering problem must look like far away from the source of scattering. Express the differential cross section in terms of the general solution of the wave function. (**4p**)

3. Determine, using the first Born approximation in the two dimensional case defined above, the differential and the total scattering cross-section in the low energy limit for a spherical potential well

$$V = \begin{cases} |V_0| & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a. \end{cases}$$

(**5p**)

4. Develop the partial wave method for the two dimensional scattering problem discussed above. First solve the free Schrödinger equation to find out the proper basis function that should be used in the two dimensional case. Using these basis functions, write the plane wave $\frac{1}{(2\pi)}e^{ikx}$ in polar coordinates. (**5p**)

5. Using the partial wave method applied to the two dimensional problem show that the cross section for scattering on a hard sphere of radius R (well, it should be called a hard disc in the two dimensional case) is given by the formula

$$\sigma = \frac{4}{k} \sum_{n=-\infty}^{\infty} \frac{J_n^2(kR)}{J_n^2(kR) + N_n^2(kR)}$$

where $k = \sqrt{2mE}/\hbar$. What is the total cross section in the low energy limit? (**5p**)

Useful formulas that may be used without further proof.

$$\begin{aligned} \int \frac{d^2 l}{(2\pi)^2} \frac{e^{i\bar{l}\cdot\bar{x}}}{E - \frac{\hbar^2 \bar{l}^2}{2m} + i\epsilon} &= -\frac{m}{\pi\hbar^2} K_0\left(-ik|\bar{x}|\right) \ , \quad E = \frac{\hbar^2 k^2}{2m} \\ \lim_{|z|\to\infty} K_0(z) &= \sqrt{\frac{\pi}{2z}} e^{-z} \end{aligned}$$

where K_0 is a so called modified Bessel function.
The Bessel functions J_n and N_n are the two linearly independent solutions to the differential equation:

$$\frac{d^2\psi}{dz^2} + \frac{1}{z}\frac{d\psi}{dz} + \left(1 - \frac{n^2}{z^2}\right)\psi = 0$$

Here are some asymptotic formulas for ordinary Bessel functions:

$$\begin{aligned}
\lim_{z\to 0} J_n(z) &= \frac{1}{n!}\left(\frac{z}{2}\right)^n \\
\lim_{z\to 0} N_0(z) &= \frac{2}{\pi}\ln\left(\frac{z}{2}\right) \\
\lim_{z\to 0} N_n(z) &= -\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^n \\
\lim_{z\to \infty} J_n(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z-(2n+1)\frac{\pi}{4}\right) \\
\lim_{z\to \infty} N_n(z) &= \sqrt{\frac{2}{\pi z}}\sin\left(z-(2n+1)\frac{\pi}{4}\right)
\end{aligned}$$

and some other useful relations

$$\begin{aligned}
J_{-n}(z) &= (-1)^n J_n(z) \\
N_{-n}(z) &= (-1)^n N_n(z) \\
\int_0^\pi d\theta e^{ia\cos\theta} &= \pi J_0(a) \\
\int_0^1 dx x J_0(ax) &= \frac{1}{a}J_1(a)
\end{aligned}$$