Hand-in assignments in Advanced Quantum Mechanics, spring semester 2025.

These are hand in assignments for the course in Advanced Quantum mechanics at the Masaryk University in the spring of year 2025. They are the first part of the requirement of the course, the second being an oral exam. The problems should be handed in minimum one week before the oral exam. **Do not leave out any part of the calculations and motivate your assumptions and approximations carefully**. You my answer in Czech *or* English. The required minimum number of points is **22** evenly distributed over the different topics.

The formalism

1. Imagine that you are given a nonorthonormal basis $|i\rangle$ so that $B_{ik} = \langle i|k\rangle$ is a general invertible matrix. Use B to write an expression for the unity operator in this basis. Write an arbitrary state $|\psi\rangle$ in this basis, *i.e.* find an expression for c_k in

   $$|\psi\rangle = \sum_k |k\rangle \, c_k$$

   Also, if A is any operator, find an expression for the representation of this operator in the given basis

   $$A = \sum_{k,l} |k\rangle \, A_{kl} \, \langle l|$$

   (4p)

2. For the one dimensional harmonic oscillator with the basis $|n\rangle$ and ladder operators

   $$\begin{aligned} \hat{a}\ ,\ \ \hat{a}^\dagger \quad &: \quad [\hat{a}, \hat{a}^\dagger] = 1 \\ \hat{a}\,|n\rangle \quad &= \quad \sqrt{n}\,|n-1\rangle\ ,\ \ \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{aligned}$$

   define the *Coherent states* $|z\rangle$ as eigenstates (with complex eigenvalue z) of the annihilation operator

   $$\hat{a}\,|z\rangle = z\,|z\rangle$$

Show that they can be written in the original basis as

$$|z\rangle = e^{z\hat{a}^\dagger - \bar{z}\hat{a}}\,|0\rangle$$

and that they are normalized but *not* orthogonal to each other by proving that

$$\langle z_1|z_2\rangle = e^{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + \bar{z}_1 z_2}$$

What conditions should the function $f(z,\bar{z})$ fulfill in order for

$$\hat{\rho} = \int d^2z f(z,\bar{z})|z\rangle\langle z|$$

to be a density operator? If $f(z,\bar{z}) = \frac{1}{\pi}e^{-|z|^2}$, is the density operator pure or mixed? For this choice of $f(z,\bar{z})$, calculate the expectation value of the operator $\hat{a}^\dagger\hat{a}$. **(4p)**

Propagators and Path Integrals

1. Here we will calculate the propagator for a harmonic oscillator in the coherent state basis introduced before. Use the coherent states defined as

   $$|z\rangle = e^{z\hat{a}^\dagger - \bar{z}\hat{a}}\,|0\rangle\ .$$

   In order to do this it is useful to prove that

   $$e^{-i\omega t\,\hat{a}^\dagger\hat{a}}(z\hat{a}^\dagger - \bar{z}\hat{a})e^{i\omega t\,\hat{a}^\dagger\hat{a}} = z(t)\hat{a}^\dagger - \bar{z}(t)\hat{a}$$

   where

   $$z(t) = e^{-i\omega\,t}z$$

   Calculate the propagator for the harmonic oscillator

   $$\langle z_f|\,\hat{U}(t,0)\,|z_i\rangle\ .$$

   where $|z_i\rangle$ is the initial state at time $t=0$ and $|z_f\rangle$ is the final state at time t. *Interpret* your result.**(4p)**

2. Find the time evolution of the 1 dimensional harmonic oscillator state

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega(x-x_0)^2}{2\hbar}},$$

using the harmonic oscillator propagator

$$\begin{aligned} K(x',t';x,t) &= \sqrt{\frac{m\omega}{2\pi i\hbar \sin\omega\Delta t}} \times \\ &\exp\left\{\left(\frac{im\omega}{2\hbar\sin\omega\Delta t}\right)\left((x'^2+x^2)\cos\omega\Delta t - 2x'x\right)\right\} \end{aligned}$$

Interpret your result! (**4p**)

Relativistic quantum mechanics

1. If we interpret $\rho = \psi^\dagger\psi$ as the particle density, use the Dirac Hamiltonian to define a current $\mathbf{j}$ such that the equation of continuity is fulfilled

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0$$

In order to do this it is useful to have the relation

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

which you should prove. Calculate ρ and $\mathbf{j}$ for the plane wave $\psi(\mathbf{x}) = u(\mathbf{p})e^{-\frac{i}{\hbar}Et}e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$ normalized so that $\psi^\dagger\gamma^0\psi = 1$. Comment on your result. **(4p)**

2. Prove the following trace identities valid for any representation

$$\begin{aligned} \mathrm{tr}(\gamma^\mu) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu) &= 4\eta^{\mu\nu} \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) &= 0 \\ \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \end{aligned}$$

using *only* the anticommutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and the properties of the trace. To prove the first and the third relation, it is useful to define the matrix

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^2$$

and show that $\gamma_5\gamma_5 = 1$ and that it anticommutes with all the γ^μ. **(4p)**

3. Define the spin operator

$$\Sigma_k = \frac{1}{2i}\epsilon_{klm}\gamma^l\gamma^m$$

and find the eigenvectors and eigenvalues of Σ_3. Show that it does not commute with the Dirac Hamiltonian in general so it is not a conserved quantity. However, if we measure the spin along the direction of motion we may define the helicity operator

$$\frac{\mathbf{p}\cdot\mathbf{\Sigma}}{|\mathbf{p}|}$$

Show that it commutes with the Dirac Hamiltonian and thus that its value is conserved during time evolution. **(4p)**

Scattering theory

1. Imagine studying particle scattering in a two dimensional world (there are plenty of examples of effectively two dimensional physical systems in Condensed Matter Physics). How would one define the cross-section? What dimension (unit) would it have? (**2p**)

2. Show how the wavefunction in the 2D scattering problem must look like far away from the source of scattering. Express the differential cross section in terms of the general solution of the wave function. (**4p**)

3. Determine, using the first Born approximation in the two dimensional case defined above, the differential and the total scattering cross-section in the low energy limit for a spherical potential well

$$V = \begin{cases} |V_0| & \text{for} \quad r < a \\ 0 & \text{for} \quad r > a. \end{cases}$$

(**5p**)

4. Develop the partial wave method for the two dimensional scattering problem discussed above. First solve the free Schrödinger equation to find out the proper basis function that should be used in the two dimensional case. Using these basis functions, write the plane wave $\frac{1}{(2\pi)}e^{ikx}$ in polar coordinates. What is the total cross section on a "hard disc"? (**5p**)

Useful formulas that may be used without further proof.

$$
\begin{aligned}
\int \frac{d^2 l}{(2\pi)^2} \frac{e^{i\bar{l}\cdot\bar{x}}}{E - \frac{\hbar^2 \bar{l}^2}{2m} + i\epsilon} &= -\frac{m}{\pi\hbar^2} K_0\left(-ik|\bar{x}|\right) \ , \quad E = \frac{\hbar^2 k^2}{2m} \\
\lim_{|z|\to\infty} K_0(z) &= \sqrt{\frac{\pi}{2z}} e^{-z}
\end{aligned}
$$

where K_0 is a so called modified Bessel function.
The Bessel functions J_n and N_n are the two linearly independent solutions to the differential equation:

$$
\frac{d^2\psi}{dz^2} + \frac{1}{z}\frac{d\psi}{dz} + \left(1 - \frac{n^2}{z^2}\right)\psi = 0
$$

Here are some asymptotic formulas for ordinary Bessel functions:

$$
\begin{aligned}
\lim_{z\to 0} J_n(z) &= \frac{1}{n!}\left(\frac{z}{2}\right)^n \\
\lim_{z\to 0} N_0(z) &= \frac{2}{\pi}\ln\left(\frac{z}{2}\right) \\
\lim_{z\to 0} N_n(z) &= -\frac{(n-1)!}{\pi}\left(\frac{2}{z}\right)^n \\
\lim_{z\to\infty} J_n(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z - (2n+1)\frac{\pi}{4}\right) \\
\lim_{z\to\infty} N_n(z) &= \sqrt{\frac{2}{\pi z}}\sin\left(z - (2n+1)\frac{\pi}{4}\right)
\end{aligned}
$$

and some other useful relations

$$
\begin{aligned}
J_{-n}(z) &= (-1)^n J_n(z) \\
N_{-n}(z) &= (-1)^n N_n(z) \\
\int_0^\pi d\theta e^{ia\cos\theta} &= \pi J_0(a) \\
\int_0^1 dx x J_0(ax) &= \frac{1}{a} J_1(a)
\end{aligned}
$$