Useful relations

We are working in a metric with a "mostly minus" or "west-cost" signature $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. The gamma matrices are 4×4 matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \ i = 1, 2, 3$$

where σ_i are the Pauli matrices. It is easy to check that $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The Dirac equation in momentum space can be written $(\not{k} - m)u = 0$ where $\not{k} = \gamma^\mu k_\mu$. There are 4 independent solutions which can be written as

$$u^{(1)} = \sqrt{E+m} \begin{pmatrix} 1 \\ 0 \\ \frac{p_3}{E+m} \\ \frac{p_+}{E+m} \end{pmatrix} u^{(2)} = \sqrt{E+m} \begin{pmatrix} 0 \\ 1 \\ \frac{p_-}{E+m} \\ -\frac{p_3}{E+m} \end{pmatrix}$$

$$v^{(1)} = \sqrt{|E|+m} \begin{pmatrix} -\frac{p_-}{|E|+m} \\ \frac{p_3}{|E|+m} \\ 0 \\ -1 \end{pmatrix} v^{(2)} = \sqrt{|E|+m} \begin{pmatrix} \frac{p_3}{|E|+m} \\ \frac{p_+}{|E|+m} \\ 1 \\ 0 \end{pmatrix}$$

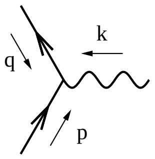
where $p_\pm = p_1 \pm ip_2, E = \sqrt{p^2 + m^2}$ and the first two solutions represents spin up/down electrons and the last two solutions represents spin up/down positrons. Using these solutions it is possible to show that $u_a^{(1)}\bar{u}_b^{(1)} + u_a^{(2)}\bar{u}_b^{(2)} = (\not{p} + m)_{ab}$ and $v_a^{(1)}\bar{v}_b^{(1)} + v_a^{(2)}\bar{v}_b^{(2)} = (\not{p} - m)_{ab}$

Here are some useful gamma matrix relations:

$$\begin{aligned}
\mathrm{Tr}\,(\gamma^\mu) &= \mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\sigma) = \mathrm{Tr}\,\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\gamma^\delta\right) = \ldots = 0 \\
\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{Tr}\,\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) &= 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right) \\
\gamma_\mu\gamma^\mu &= 4 \\
\gamma_\mu\gamma^\nu\gamma^\mu &= -2\gamma^\nu \\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu &= 4g^{\alpha\beta} \\
\gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\mu &= -2\gamma^\gamma\gamma^\beta\gamma^\alpha
\end{aligned}$$

Feynman rules

- Do everything in momentum space.
- The basic vertices can be found from the interaction Hamiltonian. In pure QED we have $-ie\int d^4x\hat{\bar{\psi}}\hat{\not{A}}\hat{\psi}$ so the basic vertex can be drawn as



and it comes with a factor $-ie\gamma^\mu(2\pi)^4\delta^4(p+q+k)$.

- Since each vertex has a factor of the electric charge e which is small ($e^2 = \frac{1}{137}$) the calculation can be done as an expansion in the number of vertices. That is, for a given process, draw all possible (connected) diagrams that can contribute and that contains less than a certain fixed number of vertices.

- Be careful when counting the number of vertices. If you are drawing a diagram with n vertices there is a factor of $\frac{1}{n!}$ from the expansion of the $T\exp(-i\int H)$ operator. Then one draws all possible diagrams but some of them are the same because the integration over the positions of the vertices.

- Here one should also be careful about relative signs between different diagrams coming from anticommutation of fermionic operators.

- Every outgoing electron comes with a wavefunction factor of $\frac{\bar{u}(p)}{\sqrt{2VE(p)}}$. Every incoming electron comes with a factor of $\frac{u(p)}{\sqrt{2VE(p)}}$. Every outgoing positron comes with a factor of $\frac{v(p)}{\sqrt{2VE(p)}}$ and every incoming positron comes with a factor of $\frac{\bar{v}(p)}{\sqrt{2VE(p)}}$.

- Every outgoing photon comes with a factor of $\frac{\sqrt{4\pi}(\epsilon_\mu^{(\alpha)}(k))^*}{\sqrt{2V\omega(k)}}$ and every incoming photon comes with a factor of $\frac{\sqrt{4\pi}\epsilon_\mu^{(\alpha)}(k)}{\sqrt{2V\omega(k)}}$ where $\alpha = 1,2$ since only physical polarizations are allowed for external photons.

- Every internal electron/positron line comes with a propagator

$$i\int\frac{d^4p}{(2\pi)^4}\frac{(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon}$$

- Every internal photon line comes with a propagator

$$-i\int\frac{d^4k}{(2\pi)^4}\frac{4\pi g_{\mu\nu}}{k^2+i\epsilon}$$

- The easiest way to write down the expression for a diagram is to follow the electron/positron backwards in time. First you have an outgoing electron/positron and you need to write down the wavefunction (as given above). Then there is a vertex so you need the $-ie\gamma$ and the momentum conserving delta function as given above. Next is either an incoming electron/positron and you write down its wave function and you are ready, or there is a propagator which you write down as given above and then there is a new vertex which gives you a new factor of $-ie\gamma$ and so on and so on. After you have written down the electron/positron lines you fill in the photons. If they are external you get wavefunctions and if they go between two different electrons you get propagators connecting the gamma matrices.

Calculation of cross sections

- Calculate the probability amplitude for some event using the Feynman rules given above.
- To calculate the probability one has to take the absolute value squared of the probability amplitude. Since one always have a delta function expressing the conservation of total momentum $(2\pi)^4\delta^4(p_1 + \ldots + p_n)$ one will get this delta function squared in the expression for the probability. This we can rewrite, using the expression for the delta function as an integral, as

  $$\left|(2\pi)^4\delta^4(p_1 + \ldots + p_n)\right|^2 = (2\pi)^4\delta^4(p_1 + \ldots + p_n)\int d^4x e^{i(p_1+\ldots+p_n)\cdot x},$$

  but since we have the second delta function we can replace the exponential with 1 and we get

  $$\left|(2\pi)^4\delta^4(p_1 + \ldots + p_n)\right|^2 = VT(2\pi)^4\delta^4(p_1 + \ldots + p_n),$$

  where V is the volume of the universe and T is the time we let the interaction act so that $VT = \int d^4x$.
- To find the total probability we also have to notice that the we cannot experimentally separate final states which are too close in momenta. That is, if we are interested in the probability of measuring a particle in the final state which has momentum p_f, what we will measure is the probability for a final state with momentum p_f *plus* the probability for a final state with momentum $p_f + \delta p_f$ *plus* the probability for all other state with close enough momenta. Since for all these states the probability is the same we can write the total probability as (#states) $\times P(p_f)$. The number of states is easy to find since we know the density of states in phase space $\frac{V}{(2\pi)^3}$ and we arrive at the rule: for every final state we should include a factor

  $$\frac{V}{(2\pi)^3}d^3p_i$$

- Now we have calculated the probability that the process in question will take place if we have one particle of each kind in the whole universe and we measure for T (seconds, say). This is not what we want. In the usual experimental situation we are shooting a stream of some kind of particles (say photons) at a target (say an electron) and we are interested in how many of the incoming particles will get scattered per unit time (second, say). To get a number which is independent of the incoming flux we calculate the *cross section*. Mathematically we can express this as

  $$\frac{P}{T} = d\sigma \times (\text{incoming flux}),$$

  or in words: the probability per unit time is the cross section times the incoming flux (incoming flux means the number of incoming particles per unit time and unit area). From this formula we can compute the cross-section.

- The flux given by one of our states with speed v we can calculate as follows: we have to calculate how many particles pass an area A in time T. In time T a total volume of vTA will pass the area A. Since the wavefunctions of our particles are normalized to one particle in the whole universe we will count on average $\frac{vTA}{V}$ particles coming through our area A. This means that we have a flux $\frac{vTA}{VTA} = \frac{v}{V}$.

- Now we can do the integrals over some of the final momenta to get rid of the delta function. The result is the cross section for some particular polarizations of incoming and outgoing particles. If this is what one wants, then one stops here. However, often the incoming states are not polarized and in the outgoing states we are not observing the polarization. This leads to further calculations. First take the incoming states. If the incoming particles are not polarized we have no exact wave-function description. Rather we have to describe the incoming state in terms of a density matrix (reflecting the fact that we have no information about the indivitual phases of the incoming particles). This means that we have to *average* over the incoming spin states (with the correpsonding probabilities), in the half spin up half spin down case this means to include the factor $\frac{1}{2}\sum_s$. For outgoing particles there are no density matrices, the particles are in pure states. We can calculate the probability to measure spin down as well as spin up but if we in the experiment do not make a difference between these possibilities and calculate all events regardless of spin we must sum the different probabilities. The probability to measure spin up *or* spin down is the sum of the two separate probabilities. This also leads to a sum over spins. In evaluating these sums we use the formulas for summing over polarizations given in the first section.