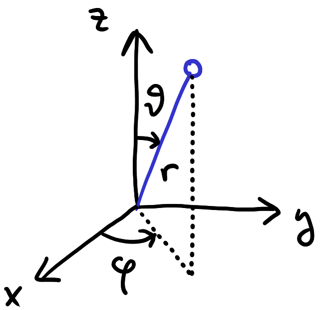
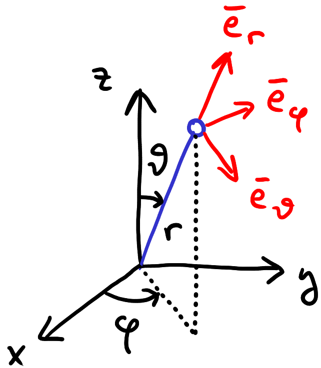
Sférické souřadnice a moment hybnosti



$$x = r \sin\vartheta \cos\varphi$$

$$y = r \sin\vartheta \sin\varphi$$

$$z = r \cos\vartheta$$



$$\bar{e}_r = \frac{\partial(x,y,z)}{\partial r} = (\sin\vartheta\cos\varphi,\ \sin\vartheta\sin\varphi,\ \cos\vartheta)$$

$$\bar{e}_\vartheta = \text{normovaný } \frac{\partial(x,y,z)}{\partial\vartheta} = (\cos\vartheta\cos\varphi,\ \cos\vartheta\sin\varphi,\ -\sin\vartheta)$$

$$\bar{e}_\varphi = (-\sin\varphi,\ \cos\varphi)$$

nabla operátor v lokální bázi $\nabla = \bar{e}_r \frac{\partial}{\partial r} + \bar{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial\vartheta} + \bar{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$

$$\hat{\bar{L}} = \hat{\bar{r}} \times \hat{\bar{p}} = \hat{\bar{r}} \times \frac{\hbar}{i}\nabla = \bar{e}_r r \times \frac{\hbar}{i}\left(\bar{e}_r \frac{\partial}{\partial r} + \bar{e}_\vartheta \frac{1}{r}\frac{\partial}{\partial\vartheta} + \bar{e}_\varphi \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)$$

$\rightarrow$ uplatní se jenom derivace podle ϑ a φ

$$\hat{\bar{L}} = -i\hbar\left(\underbrace{\bar{e}_r \times \bar{e}_\vartheta}_{\bar{e}_\varphi}\frac{\partial}{\partial\vartheta} + \underbrace{\bar{e}_r \times \bar{e}_\varphi}_{-\bar{e}_\vartheta}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right) = -i\hbar\left(\bar{e}_\varphi\frac{\partial}{\partial\vartheta} - \bar{e}_\vartheta\frac{1}{\sin\vartheta}\frac{\partial}{\partial\varphi}\right)$$

v kombinaci s $\bar{e}_\vartheta = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta)$ a $\bar{e}_\varphi = (-\sin\varphi, \cos\varphi)$

$$\left.\begin{aligned}
\hat{L}_x &= i\hbar\left(\sin\varphi\frac{\partial}{\partial\vartheta} + \operatorname{cotan}\vartheta\cos\varphi\frac{\partial}{\partial\varphi}\right)\\
\hat{L}_y &= -i\hbar\left(\cos\varphi\frac{\partial}{\partial\vartheta} - \operatorname{cotan}\vartheta\sin\varphi\frac{\partial}{\partial\varphi}\right)\\
\hat{L}_z &= -i\hbar\frac{\partial}{\partial\varphi}
\end{aligned}\right\}\quad \hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right]$$

vlastní problém se týká jen úhlových závislostí:

$$\hat{L}^2 F(\vartheta,\varphi) = \lambda F(\vartheta,\varphi) \qquad -\hbar^2\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 F}{\partial\varphi^2}\right] = \lambda F \qquad (1)$$

$$\hat{L}_z F(\vartheta,\varphi) = \zeta F(\vartheta,\varphi) \qquad -i\hbar\frac{\partial F}{\partial\varphi} = \zeta F \qquad (2)$$

Sférické harmonické funkce

řešení vlastního problému hledáno v separovaném tvaru $f(\vartheta,\varphi) = \Theta(\vartheta)\,\phi(\varphi)$

→ sada sférických harmonických funkcí

$$\hat{L}^2\, Y_{\ell m}(\vartheta,\varphi) = \hbar^2 \ell(\ell+1)\, Y_{\ell m}(\vartheta,\varphi) \qquad \ell = 0,1,2,3,\dots \qquad (1)$$

$$\hat{L}_z\, Y_{\ell m}(\vartheta,\varphi) = \hbar m\, Y_{\ell m}(\vartheta,\varphi) \qquad m = -\ell, -\ell+1, \dots, +\ell \qquad (2)$$

explicitně:

$$Y_{\ell m}(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\; P_{\ell m}(\cos\vartheta)\, e^{im\varphi} \qquad \text{(kladná } m\text{)}$$

$$Y_{\ell,-m}(\vartheta,\varphi) = (-1)^m\, Y^*_{\ell m} \qquad \text{(záporná } m\text{)}$$

Legendrovy polynomy

$$P_\ell(\xi) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\xi^\ell}(\xi^2-1)^\ell$$

přidružené Legendrovy polynomy

$$P_{\ell m}(\xi) = (1-\xi^2)^{\frac{m}{2}}\frac{d^m}{d\xi^m}P_\ell(\xi)$$

2.2 Společný vlastní problém pro $\hat{L}^2$ a L_z

- vlastní problém pro $\hat{L}^2$ a L_z s výhodným zápisem vlastních hodnot

$$\hat{L}^2 f(\vartheta,\varphi) = \hbar^2 l(l+1)\, f(\vartheta,\varphi) \qquad \hat{L}_z f(\vartheta,\varphi) = \hbar m\, f(\vartheta,\varphi) \tag{19}$$

- diferenciální rovnice vzniklé dosazením separovaného tvaru $f(\vartheta,\varphi) = \Theta(\vartheta)\Phi(\varphi)$

$$\frac{1}{\sin\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\left(\sin\vartheta\frac{\mathrm{d}}{\mathrm{d}\vartheta}\Theta\right) + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]\Theta = 0 \tag{20}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Phi + m^2\Phi = 0 \quad (\text{z rovnice s } \hat{L}^2) \qquad \frac{\mathrm{d}}{\mathrm{d}\varphi}\Phi = im\Phi \quad (\text{z rovnice s } \hat{L}_z) \tag{21}$$

Řešení rovnic (21) je snadné, $\Phi(\varphi) = \mathrm{e}^{im\varphi}$, z rovnice (20) se po substituci $\xi = \cos\vartheta$ a $P(\xi) = \Theta(\vartheta)$ stane Legendrova diferenciální rovnice, jejímž řešením jsou Legendrovy polynomy $P_l(\xi)$ (pro $m = 0$) nebo přidružené Legendrovy $P_{lm}(\xi)$ polynomy (pro $m \neq 0$):

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1-\xi^2)\frac{\mathrm{d}}{\mathrm{d}\xi}P\right] + \left[l(l+1) - \frac{m^2}{1-\xi^2}\right]P = 0 \tag{22}$$

- řešení vlastního problému – sférické harmonické funkce $Y_{lm}(\vartheta,\varphi) \sim P_{lm}(\cos\vartheta)\,\mathrm{e}^{im\varphi}$

$$\hat{L}^2 Y_{lm}(\vartheta,\varphi) = \hbar^2 l(l+1) Y_{lm}(\vartheta,\varphi) \qquad l = 0,1,2,3,\dots \tag{23}$$

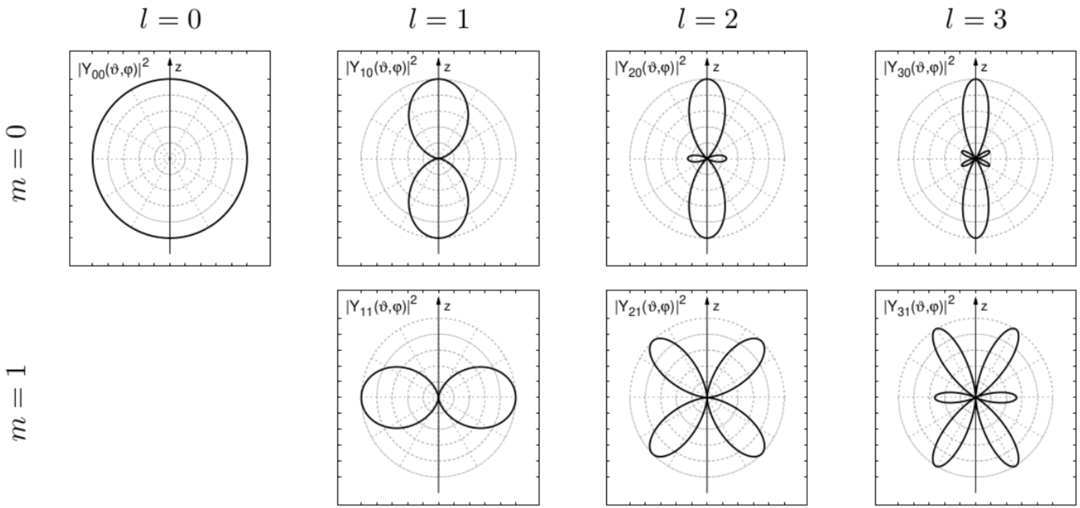
$$\hat{L}_z Y_{lm}(\vartheta,\varphi) = \hbar m Y_{lm}(\vartheta,\varphi) \qquad m = -l, -l+1, \dots, +l-1, +l \tag{24}$$

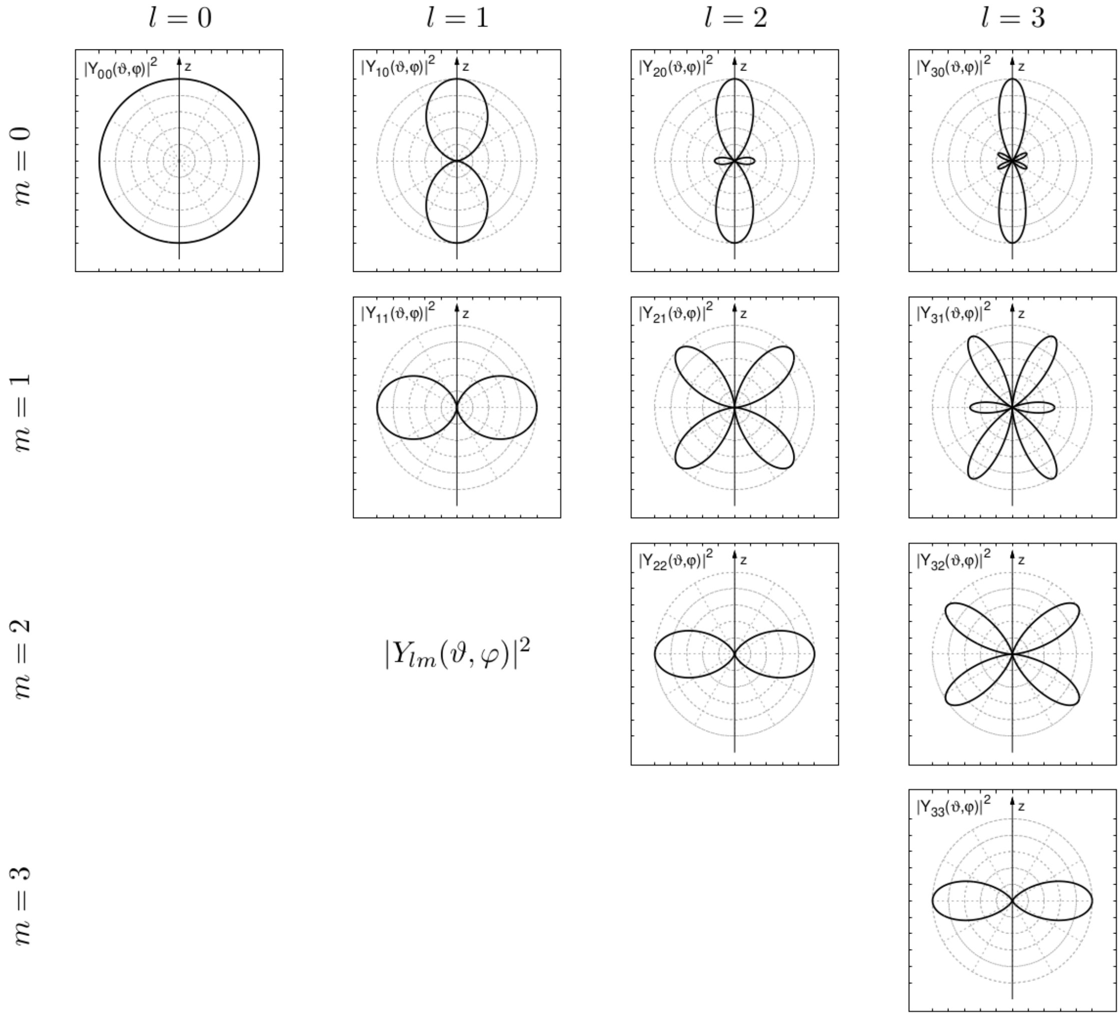
$$Y_{00} = \frac{1}{\sqrt{4\pi}} \quad Y_{10} = \sqrt{\tfrac{3}{4\pi}}\cos\vartheta \quad Y_{20} = \tfrac{1}{2}\sqrt{\tfrac{5}{4\pi}}(3\cos^2\vartheta - 1) \quad Y_{30} = \tfrac{1}{2}\sqrt{\tfrac{7}{4\pi}}(5\cos^3\vartheta - 3\cos\vartheta)$$

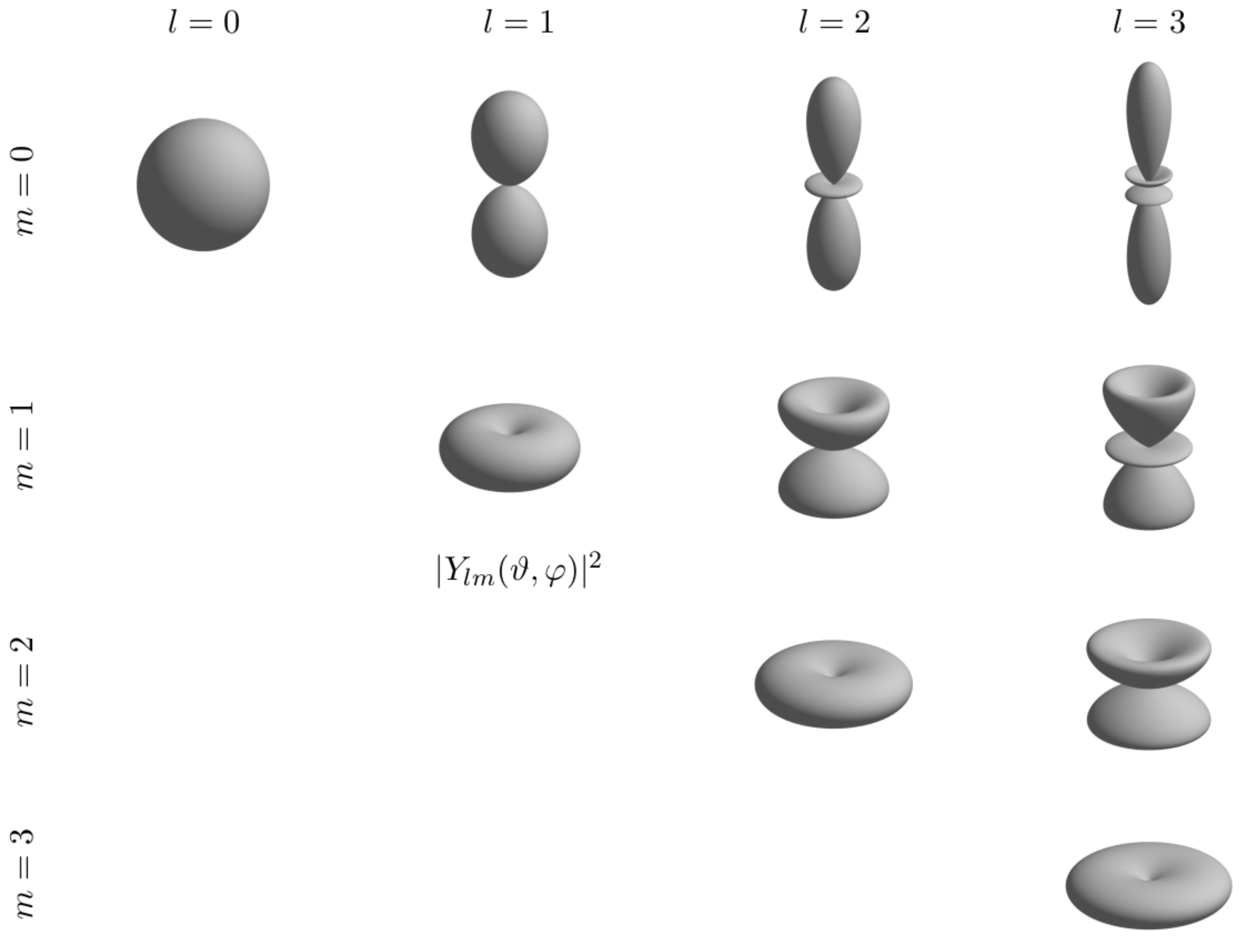
$$Y_{11} = -\sqrt{\tfrac{3}{8\pi}}\sin\vartheta\,\mathrm{e}^{\mathrm{i}\varphi} \quad Y_{21} = -\sqrt{\tfrac{15}{8\pi}}\sin\vartheta\cos\vartheta\,\mathrm{e}^{\mathrm{i}\varphi} \quad Y_{31} = -\tfrac{1}{4}\sqrt{\tfrac{21}{4\pi}}\sin\vartheta(5\cos^2\vartheta - 1)\,\mathrm{e}^{\mathrm{i}\varphi}$$

$$Y_{22} = \tfrac{1}{4}\sqrt{\tfrac{15}{2\pi}}\sin^2\vartheta\,\mathrm{e}^{2\mathrm{i}\varphi} \quad Y_{32} = \tfrac{1}{4}\sqrt{\tfrac{105}{2\pi}}\sin^2\vartheta\cos\vartheta\,\mathrm{e}^{2\mathrm{i}\varphi}$$

$$Y_{33} = -\tfrac{1}{4}\sqrt{\tfrac{35}{4\pi}}\sin^3\vartheta\,\mathrm{e}^{3\mathrm{i}\varphi}$$

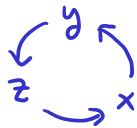



$l = 0$
$l = 1$
$l = 2$
$l = 3$
$m = 0$
$m = 1$
$m = 2$
$m = 3$
$|Y_{00}(\vartheta,\varphi)|^2$
$|Y_{10}(\vartheta,\varphi)|^2$
$|Y_{20}(\vartheta,\varphi)|^2$
$|Y_{30}(\vartheta,\varphi)|^2$
$|Y_{11}(\vartheta,\varphi)|^2$
$|Y_{21}(\vartheta,\varphi)|^2$
$|Y_{31}(\vartheta,\varphi)|^2$
$|Y_{22}(\vartheta,\varphi)|^2$
$|Y_{32}(\vartheta,\varphi)|^2$
$|Y_{33}(\vartheta,\varphi)|^2$
z
$|Y_{lm}(\vartheta,\varphi)|^2$


$l = 0$
$l = 1$
$l = 2$
$l = 3$
$m = 0$
$m = 1$
$m = 2$
$m = 3$
$|Y_{lm}(\vartheta, \varphi)|^2$

Algebraický přístup k momentu hybnosti

východiska: $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$ & cyklická záměna



spektrum vlastních stavů $\hat{L}^2, \hat{L}_z$

$$\hat{L}^2 |lm\rangle = \hbar^2 l(l+1) |lm\rangle$$

$$\hat{L}_z |lm\rangle = \hbar m |lm\rangle$$

$|0,0\rangle$ — $l = 0$

$|1,-1\rangle \quad |1,0\rangle \quad |1,+1\rangle$ — $l = 1$

$|2,-2\rangle \quad |2,-1\rangle \quad |2,0\rangle \quad |2,+1\rangle \quad |2,+2\rangle$ — $l = 2$

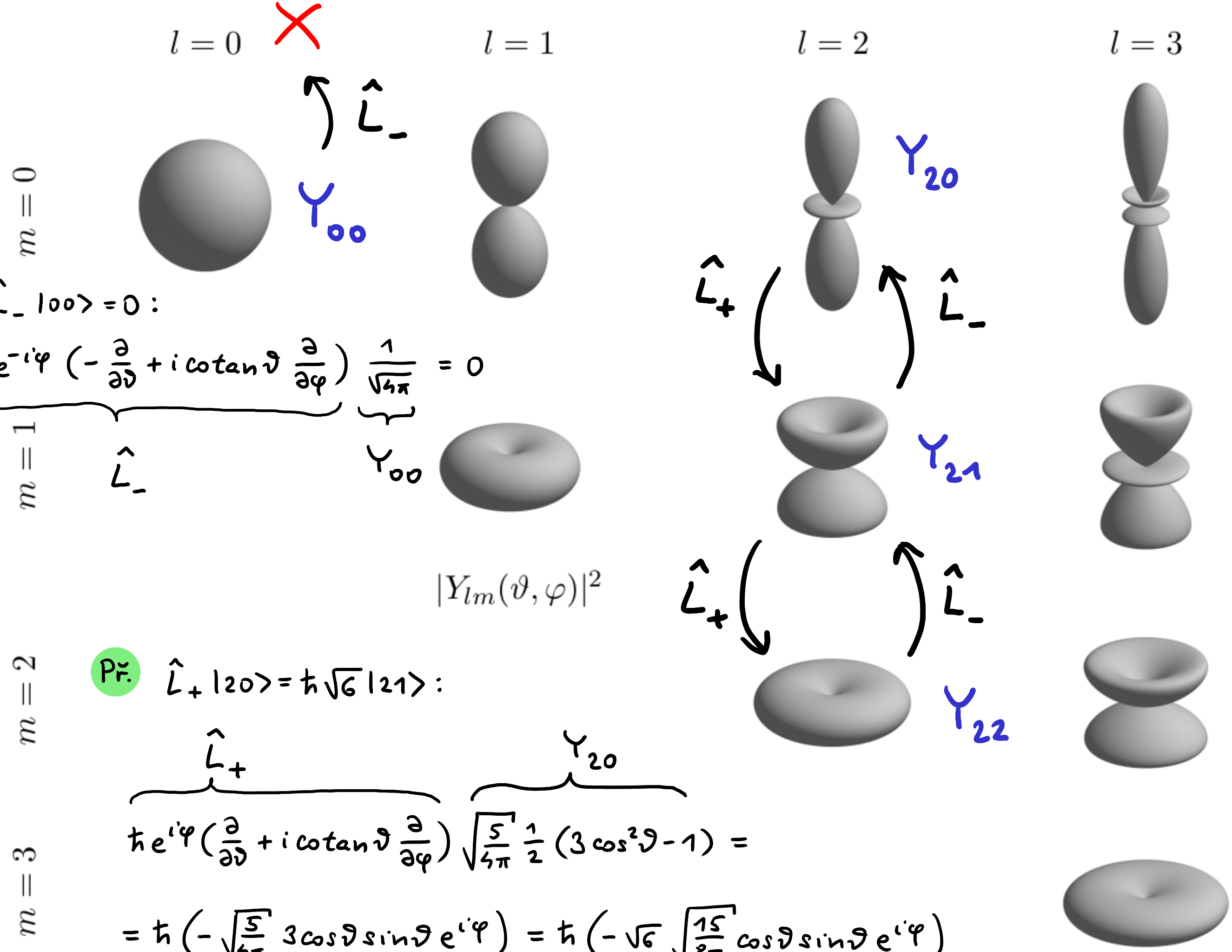
$\hat{L}_- \qquad \hat{L}_+$

žebříkové operátory

$$\hat{L}_+ = \hat{L}_x + i\hat{L}_y$$

$$\hat{L}_- = \hat{L}_x - i\hat{L}_y$$

$$\hat{L}_\pm |lm\rangle = \hbar \sqrt{l(l+1) - m(m\pm 1)}\, |l, m\pm 1\rangle$$



Pr. $\hat{L}_- |00\rangle = 0$:

$$\underbrace{\hbar e^{-i\varphi} \left(-\frac{\partial}{\partial\vartheta} + i \operatorname{cotan}\vartheta \frac{\partial}{\partial\varphi}\right)}_{\hat{L}_-} \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_{00}} = 0$$

Pr. $\hat{L}_+ |20\rangle = \hbar\sqrt{6}\,|21\rangle$:

$$\overbrace{\hbar e^{i\varphi} \left(\frac{\partial}{\partial\vartheta} + i \operatorname{cotan}\vartheta \frac{\partial}{\partial\varphi}\right)}^{\hat{L}_+} \overbrace{\sqrt{\frac{5}{4\pi}}\,\frac{1}{2}\left(3\cos^2\vartheta - 1\right)}^{Y_{20}} =$$

$$= \hbar\left(-\sqrt{\frac{5}{4\pi}}\, 3\cos\vartheta \sin\vartheta\, e^{i\varphi}\right) = \hbar\left(-\sqrt{6}\sqrt{\frac{15}{8\pi}} \cos\vartheta \sin\vartheta\, e^{i\varphi}\right)$$

Algebraický přístup k momentu hybnosti

východiska: $[\hat{J}_x, \hat{J}_y] = i\hbar\, \hat{J}_z$ & cyklická záměna

$\downarrow$

spektrum vlastních stavů $\hat{J}^2, \hat{J}_z$

$$\hat{J}^2 |jm\rangle = \hbar^2 j(j+1)\, |jm\rangle$$

$$\hat{J}_z |jm\rangle = \hbar m\, |jm\rangle$$

$|0,0\rangle$ — $j = 0$

$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \quad |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$ — $j = \tfrac{1}{2}$

$|1,-1\rangle \quad |1,0\rangle \quad |1,+1\rangle$ — $j = 1$

$|\tfrac{3}{2}, -\tfrac{3}{2}\rangle \quad |\tfrac{3}{2}, -\tfrac{1}{2}\rangle \quad |\tfrac{3}{2}, +\tfrac{1}{2}\rangle \quad |\tfrac{3}{2}, +\tfrac{3}{2}\rangle$ — $j = \tfrac{3}{2}$

$|2,-2\rangle \quad |2,-1\rangle \quad |2,0\rangle \quad |2,+1\rangle \quad |2,+2\rangle$ — $j = 2$

$$|2,0\rangle \xleftarrow{\hat{J}_-} |2,+1\rangle \xrightarrow{\hat{J}_+} |2,+2\rangle$$

žebříkové operátory

$$\hat{J}_+ = \hat{J}_x + i\,\hat{J}_y$$

$$\hat{J}_- = \hat{J}_x - i\,\hat{J}_y$$

$$\hat{J}_\pm |jm\rangle = \hbar \sqrt{j(j+1) - m(m\pm 1)}\; |j, m\pm 1\rangle$$