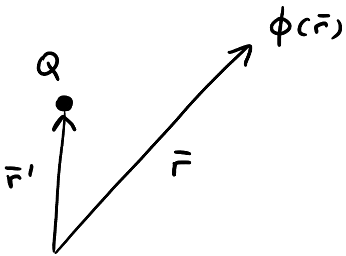
# Examples of Green's functions in classical physics

## 1 Electrostatics – electrostatic potential for a general charge distribution

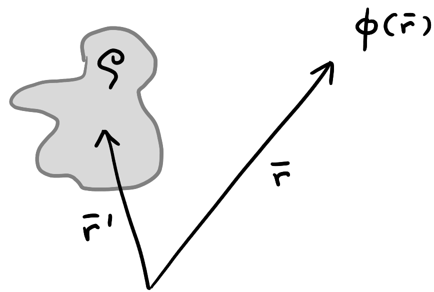
point charge




superposition principle

$\longrightarrow$

continuous charge distribution




$$\phi(\bar{r}) = \frac{Q}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$$

$$\phi(\bar{r}) = \int d^3\bar{r}' \, \frac{\rho(\bar{r}')}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|}$$

Green's Function $\quad G(\bar{r},\bar{r}') = \dfrac{1}{4\pi\varepsilon_0 |\bar{r}-\bar{r}'|} \quad \longrightarrow \quad \phi(\bar{r}) = \int d^3\bar{r} \; G(\bar{r},\bar{r}') \, \rho(\bar{r}')$

- connection to Poisson's equation $\nabla^2_{\bar r}\phi(\bar r) = -\frac{1}{\varepsilon_0}\rho(\bar r)$

$$\nabla^2_{\bar r} G(\bar r,\bar r') = -\frac{1}{\varepsilon_0}\delta(\bar r-\bar r') \qquad \rho(\bar r) = \int d^3\bar r'\,\rho(\bar r')\,\delta(\bar r-\bar r')$$

$$\int d^3\bar r'\,\rho(\bar r')\left[\nabla^2_{\bar r} G(\bar r,\bar r')\right] = \int d^3\bar r'\,\rho(\bar r')\left[-\frac{1}{\varepsilon_0}\delta(\bar r-\bar r')\right]$$

convolution $\phi(\bar r) = \int d^3\bar r\, G(\bar r,\bar r')\,\rho(\bar r')$ in short $\phi = G*\rho$

- conversion to a simple product via Fourier transform

$$\phi^{FT}(\bar q) = \int d^3\bar r\; e^{-i\bar q\cdot\bar r}\int d^3\bar r'\, G(\bar r-\bar r')\,\rho(\bar r') \qquad e^{-i\bar q\cdot\bar r} = e^{-i\bar q\cdot(\bar r-\bar r')}\,e^{-i\bar q\cdot\bar r'}$$

$$= \int d^3\bar r\int d^3\bar r'\; e^{-i\bar q\cdot(\bar r-\bar r')}\,G(\bar r-\bar r')\;e^{-i\bar q\cdot\bar r'}\rho(\bar r')$$

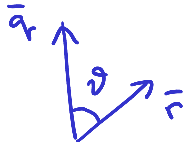
$$= G^{FT}(\bar q)\,\rho^{FT}(q)$$

Green's function in Fourier rep.: $G^{FT}(\bar q) = \dfrac{1}{\varepsilon_0 q^2}$

- explicit calculation of $G^{FT}(\bar{q})$ using real-space representation

$$G^{FT}(\bar{q}) = \int d^3\bar{r}\, e^{-i\bar{q}\cdot\bar{r}}\, G(\bar{r}) \quad \text{with} \quad G(\bar{r}) = \frac{1}{4\pi\varepsilon_0 r}$$

conversion to spherical coordinates




$\bar{q}\cdot\bar{r} = q\, r\cos\vartheta$

$$G^{FT}(q) = \int_0^\infty dr\, r^2 \underbrace{\int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi\; e^{-iqr\cos\vartheta}}_{2\pi \int_{-1}^{1} d\xi\; e^{-iqr\xi} \;=\; 2\pi \left[\frac{e^{-iqr\xi}}{-iqr}\right]_{-1}^{1}} \frac{1}{4\pi\varepsilon_0 r}$$

convergent by adding $e^{-\lambda r}$ with $\lambda \to 0^+$

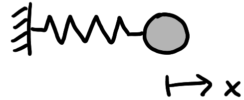
$$G^{FT}(q) = \frac{1}{-2iq\varepsilon_0} \int_0^\infty dr \left[e^{-(\lambda+iq)r} - e^{-(\lambda-iq)r}\right] =$$

$$= \frac{1}{-2iq\varepsilon_0}\left(\frac{1}{\lambda+iq} - \frac{1}{\lambda-iq}\right) = \frac{1}{\varepsilon_0}\,\frac{1}{\lambda^2+q^2} \xrightarrow{\lambda\to 0^+} \frac{1}{\varepsilon_0 q^2}$$

## 2 Green's function capturing dynamics of a classical mechanical system

- damped harmonic oscillator driven by external source

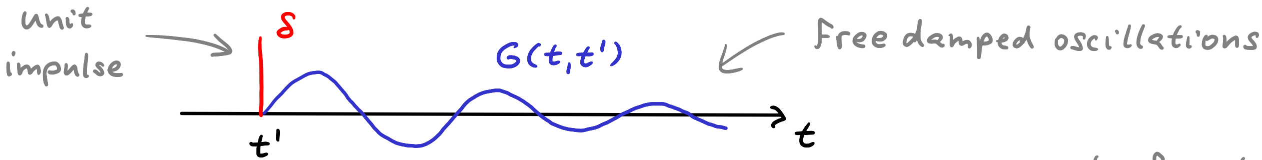
equation of motion (EOM): $m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = F(t)$




- solution via Green's function

$$F(t) = \int dt'\, F(t')\, \delta(t-t') \quad \longrightarrow \quad x(t) = \int dt'\, G(t,t')\, F(t')$$

where $G(t,t')$ is the solution of EOM with $F(t) = \delta(t-t')$




1) causality requires $G(t,t') = 0$ for $t < t'$
2) homogeneous in time $G(t,t') = G(t-t')$

step function

$$G(t) = x_0(t)\, \vartheta(t)$$

retarded Green's function

explicit calculation by inserting into EOM

$$\left.\begin{aligned} G(t) &= x_0(t)\,\vartheta(t) \\ \dot{G}(t) &= \dot{x}_0\vartheta + x_0(t=0)\,\delta \\ \ddot{G}(t) &= \ddot{x}_0\vartheta + \dot{x}_0(t=0)\,\delta + x_0(t=0)\,\dot{\delta} \end{aligned}\right\} \quad m\ddot{G} + m\gamma\dot{G} + m\omega_0^2 G =$$

$$= \underbrace{(m\ddot{x}_0 + m\gamma\dot{x}_0 + m\omega_0^2 x_0)}_{0 \text{ for all } t>0}\vartheta + \underbrace{m\dot{x}_0(t=0)}_{1}\,\delta + \underbrace{m x_0(t=0)}_{0}\,\dot{\delta} + m\gamma x_0(t=0)\,\delta = \delta$$

unit impulse on RHS

$\rightarrow$ $x_0(t)$ obeys homogeneous EOM and

initial conditions $x_0(t=0) = 0$ and $\dot{x}_0(t=0) = \frac{1}{m}$

solution: $x_0(t) = A\, e^{-\frac{\gamma t}{2}} \sin\tilde{\omega} t \qquad \tilde{\omega} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \qquad A = \frac{1}{m\tilde{\omega}}$

$$\rightarrow \quad G(t,t') = \frac{1}{m\tilde{\omega}}\, e^{-\frac{\gamma(t-t')}{2}} \sin\tilde{\omega}(t-t')\ \vartheta(t-t')$$

- Frequency domain

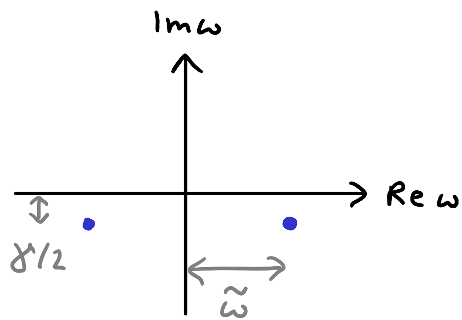
$x(t) = \int dt'\, G(t-t')\, F(t')$ convolution $\rightarrow$ simplify by Fourier transform

$x^{FT}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} x(t)$ inverse transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} x^{FT}(\omega)$

result $x^{FT}(\omega) = G^{FT}(\omega)\, F^{FT}(\omega)$ with $G^{FT}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, G(t)$

$$G^{FT}(\omega) = \dots = \frac{1}{2m\tilde{\omega}}\left(\frac{1}{\omega+\tilde{\omega}+\frac{i\gamma}{2}} - \frac{1}{\omega-\tilde{\omega}+\frac{i\gamma}{2}}\right)$$

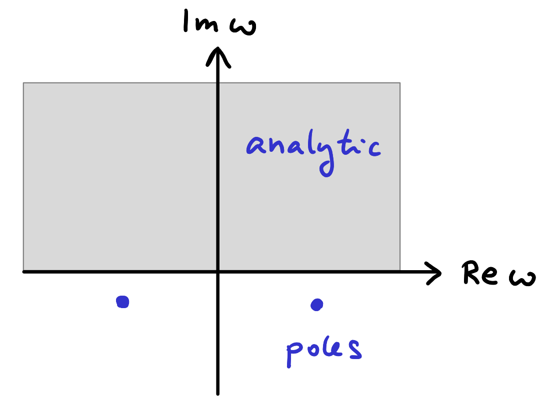
$G^{FT}(\omega)$ in complex $\omega$-plane :




poles at $\omega = \pm\tilde{\omega} - \frac{i\gamma}{2}$ (below the real axis)

- analytical structure of retarded GF

$G(\omega)$ in complex plane

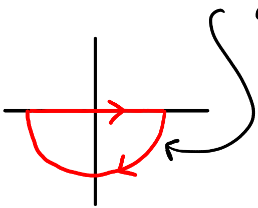



inverse transform

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\; G^{FT}(\omega)\, e^{-i\omega t}$$

evaluated by complex integration when closing the contour by an infinite semicircle

1) $t > 0$

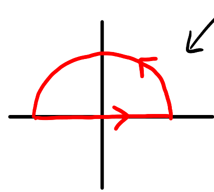


does not contribute:

$$e^{-i(\mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega)t}$$

$$\sim e^{\mathrm{Im}\,\omega\, t}$$

2) $t < 0$



vanishes: $e^{-i(\mathrm{Re}\,\omega + i\,\mathrm{Im}\,\omega)t}$

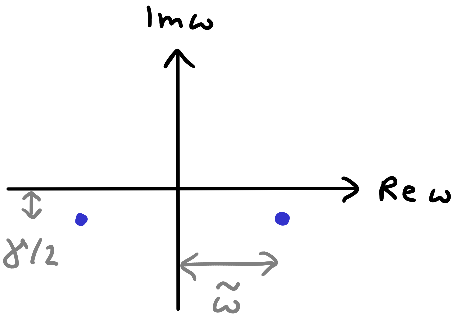
$$\sim e^{\mathrm{Im}\,\omega\, t}$$

negative

poles enclosed $\rightarrow$ $G(t>0)$ nonzero

no poles enclosed $\rightarrow$ $G(t<0) = 0$

explicit evaluation




$$G^{FT}(\omega) = \frac{1}{2m\tilde{\omega}} \left( \frac{1}{\omega + \tilde{\omega} + \frac{i\gamma}{2}} - \frac{1}{\omega - \tilde{\omega} + \frac{i\gamma}{2}} \right)$$

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \; G^{FT}(\omega)\, e^{-i\omega t}$$

$$G(t>0) = \frac{1}{2\pi} \oint d\omega \; G(\omega)\, e^{-i\omega t}$$

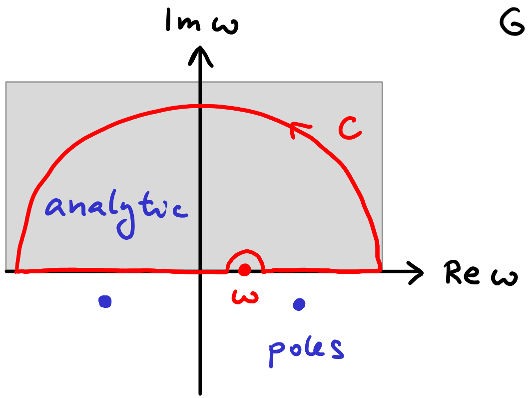
residue theorem

$$= \frac{1}{2\pi} (-2\pi i) \sum_{\text{poles}} \text{Res} \left[ G(\omega)\, e^{-i\omega t} \right]_{\omega = \text{pole}}$$

$$G(t>0) = -\frac{1}{2\pi}\, 2\pi i \, \frac{1}{2m\tilde{\omega}}\, e^{-i\left(-\tilde{\omega} - \frac{i\gamma}{2}\right)t} + \frac{1}{2\pi}\, 2\pi i\, \frac{1}{2m\tilde{\omega}}\, e^{-i\left(+\tilde{\omega} - \frac{i\gamma}{2}\right)t}$$

$$= \frac{-i}{2m\tilde{\omega}}\, e^{-\frac{\gamma t}{2}} \left( e^{i\tilde{\omega}t} - e^{-i\tilde{\omega}t} \right) = \frac{1}{m\tilde{\omega}}\, e^{-\frac{\gamma t}{2}} \sin \tilde{\omega} t$$

- causality and Kramers-Kronig relations




$G(\omega)$ analytic in upper half-plane $\leftrightarrow$ $G(t<0)=0$

$$\rightarrow \quad \oint_c d\omega' \, \frac{G(\omega')}{\omega'-\omega} = 0$$

$$\oint_c = \mathcal{P}\int_{-\infty}^{\infty} + \int + \int$$

↷ halfway around pole

$\rightarrow$ $-\frac{1}{2}$ residuum

$-\pi i \, G(\omega)$

integrand $\sim \frac{1}{z^{1+}}$

vanishes

KK relations

$$\mathcal{P}\int_{-\infty}^{\infty} d\omega' \, \frac{G(\omega')}{\omega'-\omega} = \pi i \, G(\omega)$$

$$\rightarrow \quad \mathrm{Re}\, G(\omega) = \frac{1}{\pi} \, \mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\, G(\omega')}{\omega'-\omega}$$