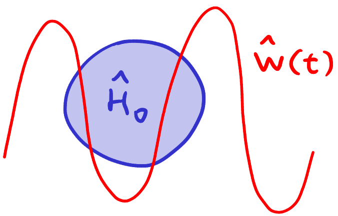
Perturbation theory & Feynman diagrams

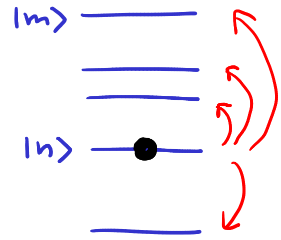
① Non-stationary perturbation theory & interaction picture



$$\hat{H}(t) = \hat{H}_0 + \hat{W}(t)$$

determines eigenstates of the unperturbed system

drives transitions between the eigenstates



- basic QM approach

eigenbasis $\hat{H}_0|n\rangle = E_n|n\rangle \quad \rightarrow \quad |\psi(t)\rangle = \sum_n c_n(t)\, e^{-\frac{i}{\hbar}E_n t}|n\rangle$

idea: c_n time-independent if $\hat{W}=0$ and weakly time-dependent for small $\hat{W}$

$i\hbar\frac{d}{dt}|\psi\rangle = [\hat{H}_0 + \hat{W}(t)]|\psi\rangle$: $\quad \sum_n \left(i\hbar\frac{dc_n}{dt} + \underline{c_n E_n\right) e^{-\frac{i}{\hbar}E_n t}}|n\rangle = \sum_n \underline{c_n e^{-\frac{i}{\hbar}E_n t}(E_n} + \hat{W})|n\rangle$

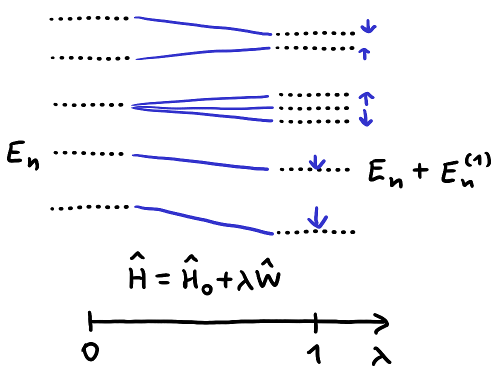
projected to $\langle m|$: $\quad i\hbar\frac{dc_m}{dt} = \sum_n \langle m|\hat{W}(t)|n\rangle\, e^{\frac{i}{\hbar}(E_m - E_n)t}\, c_n \qquad$ (still exact)

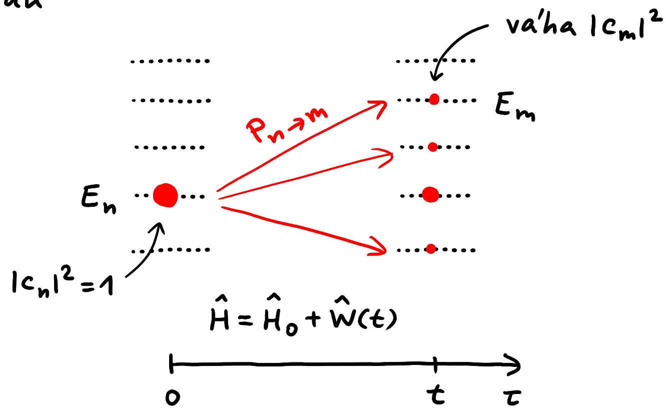
Formal integration: $$c_m(t) = c_m(0) - \frac{i}{\hbar}\int_0^t dt' \sum_n \langle m|\hat{W}(t')|n\rangle\, e^{\frac{i}{\hbar}(E_m-E_n)t'}\, c_n(t')$$

First-order PT in the case of a single eigenstate $|n\rangle$ at $t=0$:

$$c_m(0) = \delta_{mn} \quad \rightarrow \quad c_m(t) = \delta_{mn} - \frac{i}{\hbar}\int_0^t dt' \langle m|\hat{W}(t')|n\rangle\, e^{\frac{i}{\hbar}(E_m-E_n)t'}$$

- srovnání stacionární a nestacionární PT 1. řádu



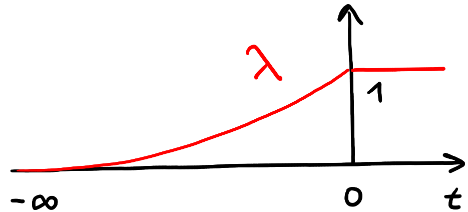


korekce vlastních energií (posuvy a štěpení) narůstající při postupném zapínání poruchy až na její plnou sílu

přechody mezi vlastními stavy $\hat{H}_0$ podnícené poruchou, charakterizováno časově závislými pravděpodobnostmi přechodů

- Is non-stationary PT able to provide energy level shifts and new eigenstates?

idea: adiabatic switching of the perturbation



$\hat{H} = \hat{H}_0 + \hat{W}\,\lambda(t)$ $\qquad \lambda(t)$ rises slowly from 0 to 1

$|\psi(t=-\infty)\rangle = |n\rangle$ eigenstate of $\hat{H}_0$

$|\psi(t=0)\rangle = |n'\rangle$ eigenstate of $\hat{H} = \hat{H}_0 + \hat{W}$

} adiabatic theorem

First order

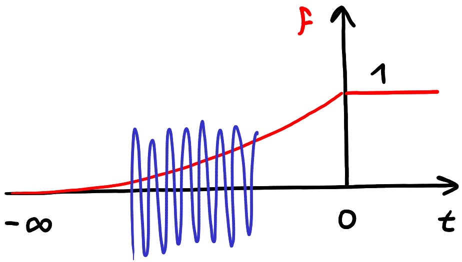
First-order energy correction

1) $m = n$ $\qquad i\hbar\,\dot{c}_n \approx \langle n|\hat{W}|n\rangle\,\lambda(t)\,c_n \longrightarrow c_n(t) = e^{-\frac{i}{\hbar}W_{nn}t}\,c_n(0) \longrightarrow \Delta E_n = W_{nn}$

2) $m \neq n$ $\qquad c_m(t=0) = -\frac{i}{\hbar}\int_{-\infty}^{0} dt'\,\langle m|\hat{W}|n\rangle\,\lambda(t')\,e^{i\frac{E_m-E_n}{\hbar}t'} = -\frac{i}{\hbar}W_{mn}\frac{\hbar}{i(E_m-E_n)} = -\frac{W_{mn}}{E_m-E_n}$

For any slowly varying $\lambda(t)$: $\int_{-\infty}^{0} dt\,\lambda(t)\,e^{i\omega t} = \frac{1}{i\omega}$

$|\psi(t=0)\rangle = |n'\rangle \approx |n\rangle - \sum_{m\neq n}\frac{W_{mn}}{E_m-E_n}|m\rangle$ $\qquad$ First-order state correction lemma: $\int\limits_{-\infty}^{0} f(t)\, e^{i\omega t}\, dt = \frac{1}{i\omega}$ for any slowly varying function $f(t)$ rising from 0 to 1



slowly varying $f(t)$

$\rightarrow$ replace $f(t)$ by its linear expansion

applied for one period of $e^{i\omega t}$

contribution of one period $[t_0, t_0 + \frac{2\pi}{\omega}]$ $\qquad f(t_0) = f_0,\ f(t+\frac{2\pi}{\omega}) = f_0 + \Delta f$

$$\int\limits_0^{2\pi/\omega} dt' \left(f_0 + \Delta f \frac{t'}{2\pi/\omega}\right) \underbrace{e^{i\omega(t_0+t')}}_{1} = \Delta f \frac{1}{2\pi} \int\limits_0^{2\pi/\omega} dt'\ \omega t'\, e^{i\omega t'} = \frac{\Delta f}{2\pi\omega} \int\limits_0^{2\pi} d\tau\ \tau\, e^{i\tau}$$

$$\overset{\text{p.p.}}{=} \frac{\Delta f}{2\pi\omega} \left\{ \left[\tau \frac{e^{i\tau}}{i}\right]_0^{2\pi} - \int\limits_0^{2\pi} d\tau \frac{e^{i\tau}}{i} \right\} = \frac{\Delta f}{2\pi\omega} \frac{2\pi}{i} = \frac{\Delta f}{i\omega}$$

$$\int\limits_{-\infty}^{0} dt\, f(t)\, e^{i\omega t} = \sum_{\text{periods}} \frac{\Delta f|_{\text{period}}}{i\omega} = \frac{1}{i\omega}$$

Further goals: 1) adapt to propagator formalism 2) infinite order

- advanced QM approach

Schrödinger picture

interaction picture / Dirac picture / Wechselwirkungsbild

$$|\psi(t)\rangle,\ \hat{H}_0 + \hat{W}(t) \longrightarrow \qquad |\tilde{\psi}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle$$

$$\tilde{W}(t) = e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{W}(t)\,e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

time evolution - states

$$i\hbar\frac{d}{dt}|\tilde{\psi}\rangle = -\hat{H}_0|\tilde{\psi}\rangle + e^{\frac{i}{\hbar}\hat{H}_0 t}\, i\hbar\frac{d}{dt}|\psi\rangle = -\hat{H}_0|\tilde{\psi}\rangle + e^{\frac{i}{\hbar}\hat{H}_0 t}\left(\hat{H}_0|\psi\rangle + \hat{W}(t)|\psi\rangle\right)$$

$$= -\hat{H}_0|\tilde{\psi}\rangle + \hat{H}_0 e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi\rangle + e^{\frac{i}{\hbar}\hat{H}_0 t}\,\hat{W}(t)\,e^{-\frac{i}{\hbar}\hat{H}_0 t}|\tilde{\psi}\rangle = \tilde{W}(t)|\tilde{\psi}\rangle$$

time evolution - operators

$$\frac{d}{dt}\tilde{A}(t) = \left(\frac{d}{dt}e^{\frac{i}{\hbar}\hat{H}_0 t}\right)\hat{A}(t)\,e^{-\frac{i}{\hbar}\hat{H}_0 t} + e^{\frac{i}{\hbar}\hat{H}_0 t}\hat{A}(t)\left(\frac{d}{dt}e^{-\frac{i}{\hbar}\hat{H}_0 t}\right) + e^{\frac{i}{\hbar}\hat{H}_0 t}\,\frac{\partial\hat{A}}{\partial t}\,e^{-\frac{i}{\hbar}\hat{H}_0 t}$$

$$= \frac{i}{\hbar}\left(\hat{H}_0\tilde{A} - \tilde{A}\hat{H}_0\right) + \frac{\partial\tilde{A}}{\partial t} = \frac{1}{i\hbar}\left[\tilde{A}, \hat{H}_0\right] + \frac{\partial\tilde{A}}{\partial t}$$

Formal integration of

$$i\hbar \frac{d}{dt} |\tilde{\psi}(t)\rangle = \tilde{W}(t) |\tilde{\psi}(t)\rangle : \qquad |\tilde{\psi}(t)\rangle = |\tilde{\psi}(0)\rangle - \frac{i}{\hbar} \int_0^t dt' \, \tilde{W}(t') |\tilde{\psi}(t')\rangle$$

First order – insert $|\tilde{\psi}(t)\rangle^{(0)} = |\tilde{\psi}(0)\rangle$ into RHS :

$$|\tilde{\psi}(t)\rangle^{(1)} = \left[\hat{1} - \frac{i}{\hbar} \int_0^t dt' \, \tilde{W}(t') \right] |\tilde{\psi}(0)\rangle$$

second order – insert $|\tilde{\psi}(t)\rangle^{(1)}$ into RHS :

$$|\tilde{\psi}(t)\rangle^{(2)} = |\tilde{\psi}(0)\rangle - \frac{i}{\hbar} \int_0^t dt_1 \, \tilde{W}(t_1) \left[|\tilde{\psi}(0)\rangle - \frac{i}{\hbar} \int_0^{t_1} dt_2 \, \tilde{W}(t_2) |\tilde{\psi}(0)\rangle \right]$$

$$= \left[\hat{1} + \left(-\frac{i}{\hbar}\right) \int_0^t dt_1 \, \tilde{W}(t_1) + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 \, \tilde{W}(t_1) \tilde{W}(t_2) \right] |\tilde{\psi}(0)\rangle$$

$\vdots$

Dyson series $\quad |\tilde{\psi}(t)\rangle = \left[\sum_{n=0}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, \tilde{W}(t_1) \tilde{W}(t_2) \cdots \tilde{W}(t_n) \right] |\tilde{\psi}(0)\rangle$

(1st version)

time-evolution operator in the interaction picture

F.J. Dyson, Phys. Rev. 75, 486 (1949)

The Radiation Theories of Tomonaga, Schwinger, and Feynman

F. J. DYSON
Institute for Advanced Study, Princeton, New Jersey

(Received October 6, 1948)

A unified development of the subject of quantum electrodynamics is outlined, embodying the main features both of the Tomonaga-Schwinger and of the Feynman radiation theory. The theory is carried to a point further than that reached by these authors, in the discussion of higher order radiative reactions and vacuum polarization phenomena. However, the theory of these higher order processes is a program rather than a definitive theory, since no general proof of the convergence of these effects is attempted.

The chief results obtained are (a) a demonstration of the equivalence of the Feynman and Schwinger theories, and (b) a considerable simplification of the procedure involved in applying the Schwinger theory to particular problems, the simplification being the greater the more complicated the problem.

I. INTRODUCTION

As a result of the recent and independent discoveries of Tomonaga,[1] Schwinger,[2] and Feynman,[3] the subject of quantum electrodynamics has made two very notable advances. On the one hand, both the foundations and the applications of the theory have been simplified by being presented in a completely relativistic way; on the other, the divergence difficulties

and ease of application, while those of Tomonaga-Schwinger are generality and theoretical completeness.

The present paper aims to show how the Schwinger theory can be applied to specific problems in such a way as to incorporate the ideas of Feynman. To make the paper reasonably self-contained it is necessary to outline the foundations of the theory, following the method

[1] Sin-itiro Tomonaga, Prog. Theoret. Phys. **1**, 27 (1946); Koba, Tati, and Tomonaga, Prog. Theoret. Phys. **2**, 101 198 (1947); S. Kanesawa and S. Tomonaga, Prog. Theoret. Phys. **3**, 1, 101 (1948); S. Tomonaga, Phys. Rev. **74**, 224 (1948).

[2] Julian Schwinger, Phys. Rev. **73**, 416 (1948); Phys. Rev. **74**, 1439 (1948). Several papers, giving a complete exposition of the theory, are in course of publication.

[3] R. P. Feynman, Rev. Mod. Phys. **20**, 367 (1948); Phys. Rev. **74**, 939, 1430 (1948); J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945). These articles describe early stages in the development of Feynman's theory, little of which is yet published.

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* Afte letter, p (1948) May 22, of radia

Expanding the product (10) in ascending powers of H_1 gives a series

$$U=1+(-i/\hbar c)\int_{-\infty}^{\sigma 0}H_1(x_1)dx_1+(-i/\hbar c)^2 \times\int_{-\infty}^{\sigma 0}dx_1\int_{-\infty}^{\sigma(x_1)}H_1(x_1)H_1(x_2)dx_2+\cdots. \quad (13)$$

As a special case of (31) obtained by replacing H^e by the unit matrix in (27),

$$S(\infty)=\sum_{n=0}^{\infty}(-i/\hbar c)^n[1/n!]\int_{-\infty}^{\infty}dx_1\cdots\int_{-\infty}^{\infty}dx_n \times P(H^I(x_1),\ \cdots,\ H^I(x_n)). \quad (32)$$

② Dyson series and thermal propagators

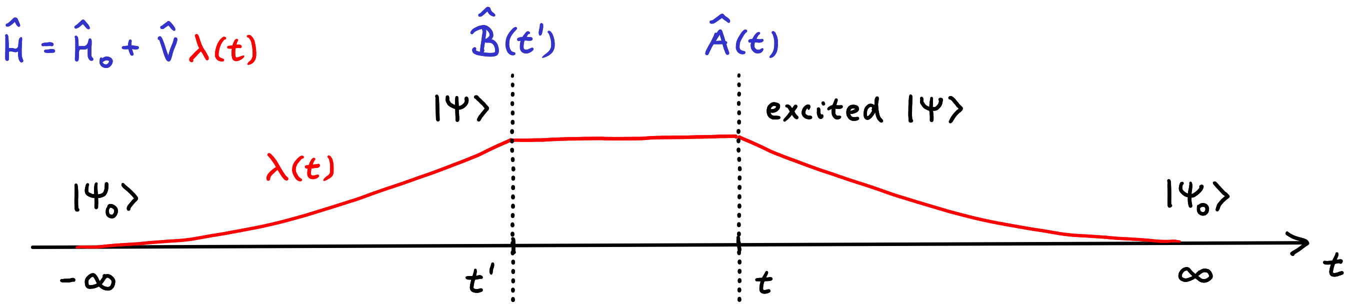
- motivation: $\hat{H} = \hat{H}_0 + \hat{V}$

$\hat{H}_0 = \sum_{k\sigma} \varepsilon_k \hat{c}^+_{k\sigma} \hat{c}_{k\sigma}$ non-interacting particles (electrons in a single band)

$\hat{V} = \frac{1}{2} \sum_{kk'q\sigma\sigma'} V_q \hat{c}^+_{k+q\sigma} \hat{c}^+_{k'-q\sigma'} \hat{c}_{k'\sigma'} \hat{c}_{k\sigma}$ pair interaction (Coulomb interaction)

start with $\hat{H}_0$ and include $\hat{V}$ perturbatively up to infinite order

- real-time propagators $G_{ret}(t,t') = -\frac{i}{\hbar} \langle [\hat{A}(t), \hat{B}(t')]_\varepsilon \rangle \vartheta(t-t')$



- time evolution operator in interaction picture

$|\tilde{\psi}(t')\rangle \rightarrow |\tilde{\psi}(t)\rangle$ using the definition $|\tilde{\psi}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle$

and time evolution in Schrödinger picture $|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$

$$|\tilde{\psi}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t}|\psi(t)\rangle = e^{\frac{i}{\hbar}\hat{H}_0 t} e^{-\frac{i}{\hbar}\hat{H}(t-t')}|\psi(t')\rangle = \underbrace{e^{\frac{i}{\hbar}\hat{H}_0 t} e^{-\frac{i}{\hbar}\hat{H}(t-t')} e^{-\frac{i}{\hbar}\hat{H}_0 t'}}_{U(t,t')}|\tilde{\psi}(t')\rangle$$

thermal version $\quad U(\tau,\tau') = e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0}$

$$\text{!} \quad U(\tau,\tau') = e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau}{\hbar}(\hat{H}_0+\hat{V})} e^{\frac{\tau'}{\hbar}(\hat{H}_0+\hat{V})} e^{-\frac{\tau'}{\hbar}\hat{H}_0} \text{ is not equal to } e^{-\frac{\tau-\tau'}{\hbar}\hat{V}} \leftarrow [\hat{H}_0,\hat{V}] \neq 0$$

properties $\quad$ 1) group property $\quad U(\tau,\tau')\,U(\tau',\tau'') = U(\tau,\tau'')$

$$\left(e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0}\right)\left(e^{\frac{\tau'}{\hbar}\hat{H}_0} e^{-\frac{\tau'-\tau''}{\hbar}\hat{H}} e^{\frac{\tau''}{\hbar}\hat{H}_0}\right) = e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau-\tau''}{\hbar}\hat{H}} e^{\frac{\tau''}{\hbar}\hat{H}_0}$$

2) reversed time evolution $\quad U(\tau',\tau) = U^{-1}(\tau,\tau')$

$$\left[e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0}\right]^{-1} = e^{\frac{\tau'}{\hbar}\hat{H}_0} e^{-\frac{\tau'-\tau}{\hbar}\hat{H}} e^{-\frac{\tau}{\hbar}\hat{H}_0}$$

- **Dyson's expansion**

assume $\tau > \tau'$, for $\tau < \tau'$ use $\mathcal{U}^{-1}(\tau';\tau)$

$$\hbar \frac{\partial \mathcal{U}(\tau,\tau')}{\partial \tau} = \hbar \frac{\partial e^{\frac{\tau}{\hbar}\hat{H}_0}}{\partial \tau} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0} + e^{\frac{\tau}{\hbar}\hat{H}_0} \hbar \frac{\partial e^{-\frac{\tau-\tau'}{\hbar}\hat{H}}}{\partial \tau} e^{-\frac{\tau'}{\hbar}\hat{H}_0}$$

$$= e^{\frac{\tau}{\hbar}\hat{H}_0} \underbrace{(\hat{H}_0 - \hat{H})}_{-\hat{V}} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0} = - \underbrace{e^{\frac{\tau}{\hbar}\hat{H}_0} \hat{V} e^{-\frac{\tau}{\hbar}\hat{H}_0}}_{\tilde{V}(\tau)} \underbrace{e^{\frac{\tau}{\hbar}\hat{H}_0} e^{-\frac{\tau-\tau'}{\hbar}\hat{H}} e^{-\frac{\tau'}{\hbar}\hat{H}_0}}_{\mathcal{U}(\tau,\tau')}$$

$\rightarrow$ $\hbar \dfrac{\partial \mathcal{U}(\tau,\tau')}{\partial \tau} = -\tilde{V}(\tau)\, \mathcal{U}(\tau,\tau')$ with initial condition $\mathcal{U}(\tau';\tau') = \hat{1}$

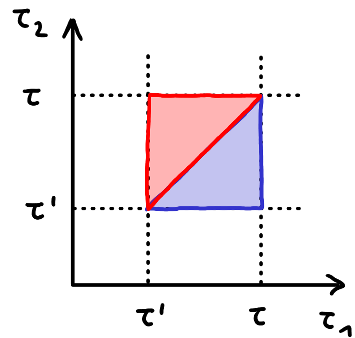
integrated $\mathcal{U}(\tau,\tau') = \hat{1} - \frac{1}{\hbar} \int_{\tau'}^{\tau} d\tau'' \, \tilde{V}(\tau'')\, \mathcal{U}(\tau'',\tau')$

solved by repeated insertion of LHS to RHS

$$\mathcal{U}(\tau,\tau') = \hat{1} - \frac{1}{\hbar} \int_{\tau'}^{\tau} d\tau_1 \, \tilde{V}(\tau_1) + \left(-\frac{1}{\hbar}\right)^2 \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau_1} d\tau_2 \, \tilde{V}(\tau_1)\tilde{V}(\tau_2) + \ldots +$$

$$+ \left(-\frac{1}{\hbar}\right)^n \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau_1} d\tau_2 \ldots \int_{\tau'}^{\tau_{n-1}} d\tau_n \, \tilde{V}(\tau_1)\tilde{V}(\tau_2) \cdots \tilde{V}(\tau_n) + \ldots$$

time-ordering trick



$$n=2: \quad \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau_1} d\tau_2\, \tilde{V}(\tau_1)\tilde{V}(\tau_2) \quad \text{equal to} \quad \int_{\tau'}^{\tau} d\tau_1 \int_{\tau_1}^{\tau} d\tau_2\, \tilde{V}(\tau_2)\tilde{V}(\tau_1)$$

$$\tau_1 \geq \tau_2 \qquad\qquad \tau_2 \geq \tau_1$$

alltogether $\quad \frac{1}{2}\int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau} d\tau_2\; T\{\tilde{V}(\tau_1)\,\tilde{V}(\tau_2)\}$ ← bosonic-like time ordering

general n: $\quad \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau_1} d\tau_2 \cdots \int_{\tau'}^{\tau_{n-1}} d\tau_n\; \tilde{V}(\tau_1)\,\tilde{V}(\tau_2)\cdots\tilde{V}(\tau_n) \qquad \tau_1 \geq \tau_2 \geq \cdots \geq \tau_n$

expressed as $\quad = \frac{1}{n!}\int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau} d\tau_2 \cdots \int_{\tau'}^{\tau} d\tau_n\; T\{\tilde{V}(\tau_1)\,\tilde{V}(\tau_2)\cdots\tilde{V}(\tau_n)\}$

Final series

$$U(\tau,\tau') = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{\hbar}\right)^n \int_{\tau'}^{\tau} d\tau_1 \int_{\tau'}^{\tau} d\tau_2 \cdots \int_{\tau'}^{\tau} d\tau_n\; T\{\tilde{V}(\tau_1)\,\tilde{V}(\tau_2)\cdots\tilde{V}(\tau_n)\} = T\, e^{-\frac{1}{\hbar}\int_{\tau'}^{\tau} d\tau''\,\tilde{V}(\tau'')}$$

- thermal propagators

$$\mathcal{G}(\tau,\tau') = -\frac{1}{\hbar}\,\frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\, T\{\hat{A}(\tau)\,\hat{B}(\tau')\}\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}$$

Heisenberg operators

$$\hat{A}(\tau) = e^{\frac{\tau}{\hbar}\hat{H}}\,\hat{A}\,e^{-\frac{\tau}{\hbar}\hat{H}}$$

interaction picture

$$\tilde{A}(\tau) = e^{\frac{\tau}{\hbar}\hat{H}_0}\,\hat{A}\,e^{-\frac{\tau}{\hbar}\hat{H}_0}$$

Heisenberg operators via interaction picture

$$\hat{A}(\tau) = e^{\frac{\tau}{\hbar}\hat{H}}\,e^{-\frac{\tau}{\hbar}\hat{H}_0}\,\tilde{A}(\tau)\,e^{\frac{\tau}{\hbar}\hat{H}_0}\,e^{-\frac{\tau}{\hbar}\hat{H}} = \mathcal{U}(0,\tau)\,\tilde{A}(\tau)\,\mathcal{U}(\tau,0)$$

Boltzmann exponential $e^{-\beta\hat{H}} = e^{-\beta\hat{H}_0}\,e^{\beta\hat{H}_0}\,e^{-\beta\hat{H}} = e^{-\beta\hat{H}_0}\,\mathcal{U}(\hbar\beta,0)$

consider only $\mathcal{G}(\tau)$ with $0\le\tau\le\hbar\beta$ and add $\tau<0$ by bosonic / fermionic symmetry relation $\mathcal{G}(\tau-\hbar\beta) = -\varepsilon\,\mathcal{G}(\tau)$, Matsubara coefficients obtained by $\int_0^{\hbar\beta} d\tau\, e^{iE_n\frac{\tau}{\hbar}}\ldots$

$\mathcal{G}(\tau\ge 0)$ numerator

$$\mathrm{Tr}\left[e^{-\beta\hat{H}_0}\,\mathcal{U}(\hbar\beta,0)\,\mathcal{U}(0,\tau)\,\tilde{A}(\tau)\,\mathcal{U}(\tau,0)\,\tilde{B}(0)\right] = \mathrm{Tr}\left[e^{-\beta\hat{H}_0}\,T\{\underline{\mathcal{U}(\hbar\beta,\tau)}\,\tilde{A}(\tau)\,\mathcal{U}(\tau,0)\,\tilde{B}(0)\}\right]$$

denominator $\mathrm{Tr}\left[e^{-\beta\hat{H}_0}\,\mathcal{U}(\hbar\beta,0)\right]$

$\hookrightarrow T\{\mathcal{U}(\hbar\beta,0)\,\tilde{A}(\tau)\,\tilde{B}(0)\}$

$\mathrm{Tr}[e^{-\beta \hat{H}_0}\text{ operators}]$ is proportional to non-interacting average $\langle \cdots \rangle_0 = \frac{1}{Z_0}\mathrm{Tr}(e^{-\beta \hat{H}_0} \ldots)$

$$\mathcal{G}(\tau) = -\frac{1}{\hbar}\frac{\mathrm{Tr}[e^{-\beta \hat{H}_0} T\{\mathcal{U}(\hbar\beta)\tilde{A}(\tau)\tilde{B}(0)\}]}{\mathrm{Tr}[e^{-\beta \hat{H}_0}\mathcal{U}(\hbar\beta,0)]} = -\frac{1}{\hbar}\frac{\langle T\{\mathcal{U}(\hbar\beta)\tilde{A}(\tau)\tilde{B}(0)\}\rangle_0}{\langle \mathcal{U}(\hbar\beta)\rangle_0}$$

analogy in $T=0$ Formalism:

non-interacting ground state

$$G(t,t') \sim \langle GS_0| T\{\mathcal{U}(\infty,t)\tilde{A}(t)\mathcal{U}(t,t')\tilde{B}(t')\mathcal{U}(t',-\infty)\}|GS_0\rangle$$

- Final expression for the thermal propagator

$$\mathcal{G}(\tau \geq 0) = -\frac{1}{\hbar}\frac{\left\langle T\left\{\sum_{n=0}^{\infty}\left(-\frac{1}{\hbar}\right)^n\frac{1}{n!}\int_0^{\hbar\beta}d\tau_1\int_0^{\hbar\beta}d\tau_2\cdots\int_0^{\hbar\beta}d\tau_n\,\tilde{V}(\tau_1)\tilde{V}(\tau_2)\cdots\tilde{V}(\tau_n)\,\tilde{A}(\tau)\tilde{B}(0)\right\}\right\rangle_0}{\left\langle T\left\{\sum_{n=0}^{\infty}\left(-\frac{1}{\hbar}\right)^n\frac{1}{n!}\int_0^{\hbar\beta}d\tau_1\int_0^{\hbar\beta}d\tau_2\cdots\int_0^{\hbar\beta}d\tau_n\,\tilde{V}(\tau_1)\tilde{V}(\tau_2)\cdots\tilde{V}(\tau_n)\right\}\right\rangle_0}$$

$\tilde{V}, \tilde{A}, \tilde{B}$ expressed using c^+, c $\xrightarrow{\text{Wick's theorem}}$ products of non-interacting $\mathcal{G}_0$

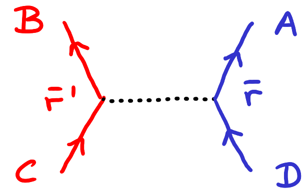
example: $V \sim c^+c^+cc \rightarrow \langle T\{(c^+c^+cc)_{\tau_1}(c^+c^+cc)_{\tau_2}\cdots\tilde{A}(\tau)\tilde{B}(0)\}\rangle_0$

③ Feynman diagrams

- pair interaction – Coulomb case

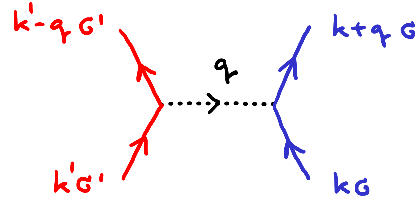
real-space representation

$$\hat V = \frac{1}{2}\sum_{\sigma\sigma'} \int d\bar r \int d\bar r'\, \underbrace{\hat\psi^+_\sigma(\bar r)}_{A}\, \underbrace{\hat\psi^+_{\sigma'}(\bar r')}_{B}\, \frac{e^2}{4\pi\varepsilon_0|\bar r-\bar r'|}\, \underbrace{\hat\psi_{\sigma'}(\bar r')}_{C}\, \underbrace{\hat\psi_\sigma(\bar r)}_{D}$$



momentum representation

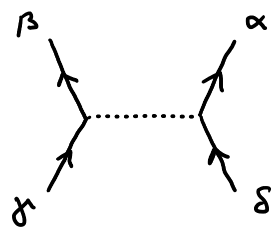
$$\hat V = \frac{1}{2}\sum_{kk'q\sigma\sigma'} V_q\, \hat c^+_{k+q\sigma}\, \hat c^+_{k'-q\sigma'}\, \hat c_{k'\sigma'}\, \hat c_{k\sigma} \quad \text{with} \quad V_q = \frac{1}{\Omega}\,\frac{1}{\varepsilon_0 q^2}$$



- general pair interactions

$$\tilde V(\tau) = \frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, \hat c^+_\alpha(\tau^+)\, \hat c^+_\beta(\tau^+)\, \hat c_\gamma(\tau)\, \hat c_\delta(\tau)$$

labels of single-particle states



- expansion of single-particle propagator modified by pair interactions

bare propagator $\mathcal{G}_0(k,\tau) = -\frac{1}{\hbar}\langle T\{\hat{c}_{k\sigma}(\tau)\,\hat{c}^+_{k\sigma}\}\rangle_0 \rightarrow \mathcal{G}_0(k,iE_n) = \frac{1}{iE_n - \varepsilon_k}$

renormalized propagator $\mathcal{G}(k,\tau) = -\frac{1}{\hbar}\langle T\{\hat{c}_{k\sigma}(\tau)\,\hat{c}^+_{k\sigma}\}\rangle = -\frac{1}{\hbar}\langle T\{\hat{c}_A(\tau)\,\hat{c}^+_B(0)\}\rangle$

Dyson expansion of the thermal propagator

$$\mathcal{G}(\tau\geq 0) = -\frac{1}{\hbar}\frac{\langle T\{\mathcal{U}(\hbar\beta)\,\hat{c}_A(\tau)\,\hat{c}^+_B(0)\}\rangle_0}{\langle \mathcal{U}(\hbar\beta)\rangle_0} \qquad \mathcal{U}(\hbar\beta) = \hat{1} + \frac{1}{1!}\left(-\frac{1}{\hbar}\right)^1 \int_0^{\hbar\beta} d\tau_1\, \tilde{V}(\tau_1) + \ldots$$

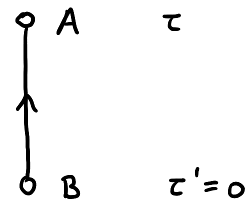
$n=0$ denominator : $\langle \mathcal{U}(\hbar\beta)\rangle_0 = 1 + \mathcal{O}(\tilde{V})$

$n=0$ numerator $\times -\frac{1}{\hbar}$:

$$-\frac{1}{\hbar}\langle T\{\mathcal{U}(\hbar\beta)\,\hat{c}_A(\tau)\,\hat{c}^+_B(0)\}\rangle_0$$

$$= -\frac{1}{\hbar}\langle T\{\hat{c}_A(\tau)\,\hat{c}^+_B(0)\}\rangle_0 + \mathcal{O}(\tilde{V})$$

$\searrow$ $\nearrow$ $\mathcal{G}_0(\tau)$

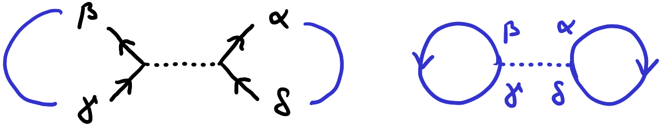


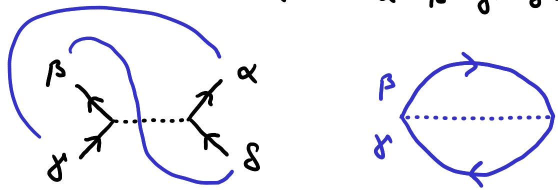
bare propagator

$$\mathcal{U}(\hbar\beta) = \hat{1} - \frac{1}{\hbar}\int_0^{\hbar\beta} d\tau_1\, \tilde{V}(\tau_1) + \dots = \hat{1} - \frac{1}{\hbar}\int_0^{\hbar\beta} d\tau_1\, \frac{1}{2}\sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta}\, \hat{c}^+_\alpha(\tau_1^+)\, \hat{c}^+_\beta(\tau_1^+)\, \hat{c}_\gamma(\tau_1)\, \hat{c}_\delta(\tau_1) + \dots$$

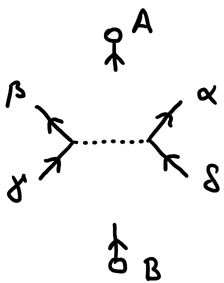
$n=1$ in the denominator

$$\langle T\{ c^+_\alpha c^+_\beta c_\gamma c_\delta \}\rangle_0 \qquad\qquad \langle T\{ c^+_\alpha c^+_\beta c_\gamma c_\delta \}\rangle_0$$





$n=1$ in the numerator : $\langle T\{ \hat{c}_A(\tau)\, \hat{c}^+_\alpha(\tau_1^+)\, \hat{c}^+_\beta(\tau_1^+)\, \hat{c}_\gamma(\tau_1)\, \hat{c}_\delta(\tau_1)\, \hat{c}^+_B(0) \}\rangle_0$



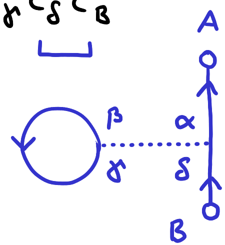
τ

τ_1

$\tau' = 0$

1) Hartree terms

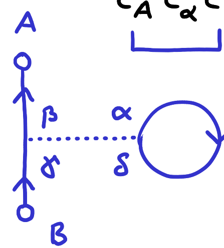
$$c_A c^+_\alpha c^+_\beta c_\gamma c_\delta c^+_B \qquad\qquad c_A c^+_\alpha c^+_\beta c_\gamma c_\delta c^+_B$$



τ

τ_1

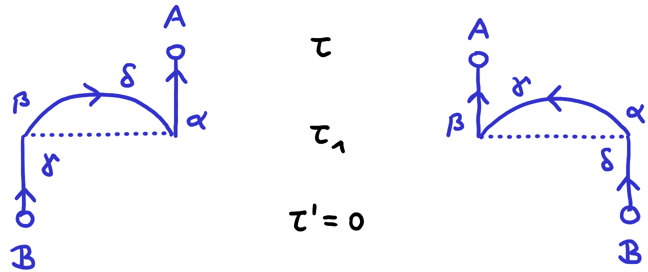
$\tau' = 0$



$n=1$ in the numerator (continued) : $\langle T \{ \hat{c}_A(\tau)\, \hat{c}^+_\alpha(\tau_1^+)\, \hat{c}^+_\beta(\tau_1^+)\, \hat{c}_\gamma(\tau_1)\, \hat{c}_\delta(\tau_1)\, \hat{c}^+_B(0) \} \rangle_0$

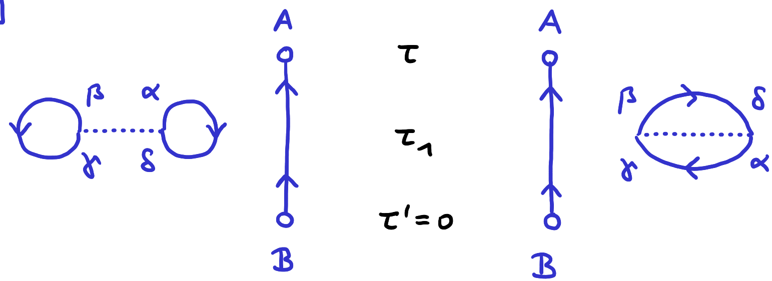
2) Hartree - Fock terms

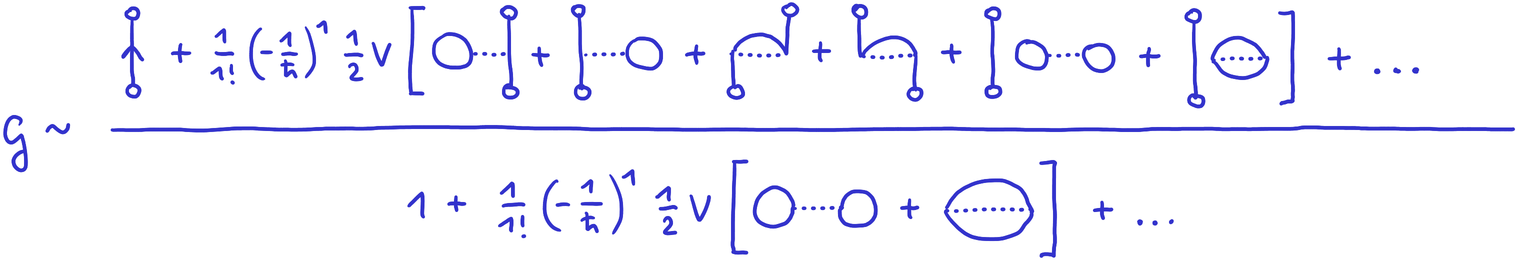
$c_A\, c^+_\alpha\, c^+_\beta\, c_\gamma\, c_\delta\, c^+_B$ $\qquad\qquad$ $c_A\, c^+_\alpha\, c^+_\beta\, c_\gamma\, c_\delta\, c^+_B$



3) disconnected terms

$c_A\, c^+_\alpha\, c^+_\beta\, c_\gamma\, c_\delta\, c^+_B$ $\qquad\qquad$ $c_A\, c^+_\alpha\, c^+_\beta\, c_\gamma\, c_\delta\, c^+_B$





general n: denominator $2n \times \hat{c}^+$, $2n \times \hat{c}$ $\rightarrow$ $2n!$ terms

numerator $2n+1 \times \hat{c}^+$, $2n+1 \times \hat{c}$ $\rightarrow$ $(2n+1)!$ terms

$\rightarrow$ Combinatorial explosion of terms

$n = 0$	1	2	3	4	5	...
1	6	120	5040	362880	39916800	...
1	2	24	720	40320	3628800	...

- reduction via:

  symmetry / topological equivalence

  cancellation of disconnected terms