FK110 Diagrammatic methods in modern condensed matter physics

Exam problem 1 – Lindhard function for a free-electron gas

Calculate the Lindhard function for a gas of non-interacting electrons with the usual free-electron dispersion

$$\varepsilon_{\boldsymbol{k}} = \frac{\hbar^2 k^2}{2m} - E_F$$

in three dimensions. Express the resulting real and imaginary part of the Lindhard function

$$\Pi(\boldsymbol{q}, E) = \frac{2}{\Omega} \sum_{\boldsymbol{k}} \frac{n_F(\varepsilon_{\boldsymbol{k}}) - n_F(\varepsilon_{\boldsymbol{k}+\boldsymbol{q}})}{\varepsilon_{\boldsymbol{k}+\boldsymbol{q}} - \varepsilon_{\boldsymbol{k}} - E - i0^+}$$

using the reduced quantities $Q = q/2k_F$ and $W = E/4E_F$. Show that the lowest term in the expansion of the reduced $\widetilde{\Pi} = (2\pi^2\hbar^2/mk_F)\,\Pi$ above the $W = Q + Q^2$ line reads as $\widetilde{\Pi}(Q, W) \approx -\frac{2}{3}(Q/W)^2$. In the RPA approximation, this result gives the plasmon energy in the $\boldsymbol{q} \to 0$ limit.

Exam problem 2 – System of coupled harmonic oscillators

Consider a system of a bosonic particle a coupled to a large set of bosonic modes b_n, the coupled system being described by the Hamiltonian

$$\mathcal{H} = \hbar\Omega\, a^\dagger a + \sum_n \hbar\omega_n\, b_n^\dagger b_n + \sum_n g_n (a^\dagger b_n + b_n^\dagger a).$$

Using the equation of motion technique, derive a set of differential equations for the thermal Green's functions related to the a and b_n particles. Note that the form of the Hamiltonian generates also off-diagonal Green's functions that need to be included. Perform Fourier transform to Matsubara representation that converts the differential equations into algebraic form and solve this set for the Green's function of the particle a. Analytically continue the result to obtain its retarded form and the corresponding spectral function. Assuming $N \to \infty$ modes b_n homogeneously covering the interval $\hbar\omega_{\min}$ to $\hbar\omega_{\max}$ and constant $g_n = g/\sqrt{N}$, determine explicitly the selfenergy of the mode a and try to plot a few representative graphs of the selfenergy and the spectral function for varying relative position of Ω with respect to the $[\omega_{\min}, \omega_{\max}]$ interval and varying coupling strength g.

Exam problem 3 – Enhancement of AF fluctuations due to Hubbard interaction

Calculate the spin susceptibility of electrons moving in a square lattice renormalized by on-site Hubbard repulsion. The electrons are described by a tight-binding Hamiltonian including nearest-neighbor hopping with an amplitude t and second nearest-neighbor hopping with an amplitude t'

$$\mathcal{H}_{tt'} = -t \sum_{\langle ij\rangle \in \mathrm{NN}} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} - t' \sum_{\langle ij\rangle \in \mathrm{nNN}} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + \mathrm{H.c.}$$

(summation over the spin projections is implied) leading to the familiar dispersion

$$\varepsilon_{\boldsymbol{k}} = -2t(\cos k_x a + \cos k_y a) - 4t' \cos k_x a \cos k_y a.$$

The total Hamiltonian of the system

$$\mathcal{H} = \sum_{\boldsymbol{k}\sigma} (\varepsilon_{\boldsymbol{k}} - \mu)\hat{c}_{\boldsymbol{k}\sigma}^\dagger \hat{c}_{\boldsymbol{k}\sigma} + U \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}$$

includes also the Hubbard interaction term that is to be treated within RPA approximation. Using a diagrammatic RPA approach, determine the zz component $\chi_{zz}(\boldsymbol{q}, E)$ of the spin susceptibility and evaluate it numerically. The necessary Brillouin zone (BZ) summations to get the bare susceptibility can be performed by utilizing a regular grid of $\boldsymbol{k}$ points covering the BZ. Take the following values for the parameters: $t = 0.4\,\mathrm{eV}$, $t' = -t/3$, band occupation $n = 0.85$, and plot the results for a few values of U that range from zero to critical U. Your plots should show both real and imaginary parts of $\chi_{zz}(\boldsymbol{q}, E)$ as maps plotted along the conventional path $\Gamma - \mathrm{X} - \mathrm{M} - \Gamma$ in the 2D Brillouin zone of the square lattice. Here $\Gamma = (0, 0)$, $\mathrm{X} = (\pi/a, 0)$, and $\mathrm{M} = (\pi/a, \pi/a)$. Additionally, show the BZ maps of static $\chi_{zz}(\boldsymbol{q}, E = 0)$ and the Fermi surface (FS) that will help you to understand the link to FS nesting. Optionally, you can contrast the results for the above nearly half-filled case with those for small ($n \lesssim 0.5$) or large ($n \gtrsim 1.5$) band filling and/or inspect the consequences of varying the t'/t ratio.