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DIFFERENTIAL INVARIANTS OF IMMERSIONS OF MANIFOLDS WITH METRIC FIELDS*

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The problem of finding all (higher order) differential invariants of immersions $f : X \to Y$, where X and Y are manifolds endowed with metric fields is investigated. The underlying jet manifolds are introduced, and the action of the corresponding differential groups on them are analysed. It is shown that the differential invariants can be described by means of the factorization of the group action with respect to a distinguished subgroup. By the orbit reduction method a basis of first order invariants is found. In particular, this basis is used for a characterization of all first order invariant Lagrangians depending on two metrics and an immersion.

Keywords: differential invariant, immersion, orbit reduction method, variational principle, invariant Lagrangians.

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1. Introduction

The problem of constructing differential invariants of a given field with values in some manifold Y, or a collection of fields, defined on a smooth manifold, belongs to the fundamental problems of the theory of invariants and its applications. In this paper, the *orbit reduction method*, developed in [8] is applied to the problem of finding all differential invariants of immersions of manifolds endowed with metric fields.

One well-known differential invariant of this type is constructed as follows. Let X (resp. Y) be a manifold, and let h (resp. g) be a metric field on X (resp. Y). Every immersion $f : X \to Y$ induces another metric on X, f^*g, and we have a differential invariant $e(f) = 1/2(f^*g)_{ij}h^{ij}$. This differential invariant, the *energy density*, defines the energy functional

$$E(f) = \int_X e(f)(x)\sqrt{|\det(h_{ij})|(x)}\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n,$$

whose critical points are the *harmonic mappings*. A particular question arises as to what extent the general covariance condition determines the function $e(f)$.

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Let us recall basic concepts of the differential invariant theory in the context of the present work (see [4], [5], [6]). Let L^r_n be the r-th differential group of the Euclidean space $\mathbf{R}^n$. Let $\mathbf{R}^{n*} \odot \mathbf{R}^{n*}$ be the symetric tensor product, where $\mathbf{R}^{n*}$ is the dual. We denote by $\mathrm{Met}\mathbf{R}^n \subset (\mathbf{R}^{n*} \odot \mathbf{R}^{n*})$ the subset of *regular* tensors, and by $T^r_n\mathrm{Met}\mathbf{R}^n$ the manifold of r-jets with source $0 \in \mathbf{R}^n$ and target in $\mathrm{Met}\mathbf{R}^n$. Further we denote by $\mathrm{Imm}J^r_{(0,0)}(\mathbf{R}^n, \mathbf{R}^m)$ the manifold of *regular* r-jets with source $0 \in \mathbf{R}^n$ and target at $0 \in \mathbf{R}^m$.

Then we set for fixed indices n and m, $n \leq m$,

$$Q^r = T^r_n\mathrm{Met}\mathbf{R}^n \times T^r_m\mathrm{Met}\mathbf{R}^m \times \mathrm{Imm}J^{r+1}_{(0,0)}(\mathbf{R}^n, \mathbf{R}^m).$$

Q^r is endowed with a natural left action of the Lie group $L^{r+1}_n \times L^{r+1}_m$ given by

$$\begin{aligned}(L^{r+1}_n \times L^{r+1}_m) \times Q^r \ni ((j^{r+1}_0\alpha, j^{r+1}_0\gamma), (j^r_0 h, j^r_0 g, j^{r+1}_0 f)) \to \\ \to (j^r_0(j^1_0\alpha \cdot h), j^r_0(j^1_0\gamma \cdot g), j^{r+1}_0(\gamma \circ f \circ \alpha^{-1})) \in Q^r,\end{aligned}$$

where $j^1_0\alpha \cdot h : \mathbf{R}^n \to \mathrm{Met}\mathbf{R}^n$ is defined by $(j^1_0\alpha \cdot h)(x) = j^1_0(t_x\alpha t_{-\alpha^{-1}(x)}) \cdot h(\alpha^{-1}(x))$, with multiplication on the right-hand side denoting the tensor action of the group $GL_n(\mathbf{R})$. Analogously for $j^1_0\gamma \cdot g$. Note that this formula represents the transformations, induced on two metric fields and their derivatives as well as derivatives of an immersion by the coordinate transformations (at a point of $\mathbf{R}^n \times \mathbf{R}^m$) (see e.g. [6]).

Now let P be any left $(L^s_n \times L^s_m)$-manifold, where $1 \leq s \leq r+1$ (later the case $P = \mathbf{R}$ will be considered). Recall that a *differential invariant* on Q^r with values in P is any smooth mapping $\chi : Q^r \to P$ such that

$$\chi((j^{r+1}_0\alpha, j^{r+1}_0\gamma) \cdot q) = (j^s_0\alpha, j^s_0\gamma) \cdot \chi(q)$$

for all $q \in Q^r$, $j^{r+1}_0\alpha \in L^{r+1}_n$, $j^{r+1}_0\gamma \in L^{r+1}_m$.

Denote by $K^{r+1,s}_n$ the kernel of the canonical group morphism $L^{r+1}_n \to L^s_n$; $K^{r+1,s}_n$ is a nilpotent normal subgroup of L^{r+1}_n. In this paper, we give a description of the quotient $Q^r/(K^{r+1,s}_n \times K^{r+1,s}_m)$ for $r = 1$, $s = 1$. This quotient has the structure of an $(L^1_n \times L^1_m)$-manifold. Any set of coordinates on this quotient manifold can be viewed as a *basis* of differential invariants. In particular, every differential invariant $\chi : Q^1 \to P$ factorizes through an $(L^1_n \times L^1_m)$-equivariant mapping $\chi_0 : Q^1/(K^{2,1}_n \times K^{2,1}_m) \to P$.

In [11], the construction of $Q^r/(K^{r+1,s}_n \times K^{r+1,s}_m)$ is given for arbitrary r, and $s = 1$; the geometrical meaning of the quotient projections $Q^r \to Q^r/(K^{r+1,1}_n \times K^{r+1,1}_m)$, or, which is the same, of bases of differential invariants, is also discussed.

For the 1st order case, treated in this paper, the basis of differential invariants gives us a criterion for an immersion to be *affine*. In the Riemannian case, a relation with the second fundamental forms appear.

As an application, we also discuss the meaning of the differential invariants of immersions for the theory of natural Lagrange structures. In particular, the result of the factorization gives us a description of all first order natural Lagrangians depending on immersion of manifolds with metric fields.

2. Preliminaries

In this section we recall two basic theorems on orbit spaces which we apply in the orbit reduction method for computing the differential invariants. The proof of the first one can be found e.g. in [1], the proof of the second is immediate (see [5]).

THEOREM 1. *Let Q be a connected left G-manifold, $\pi : Q \to Q/G$ the quotient projection. The following two conditions are equivalent:*

1. *There exists a smooth structure on Q/G such that π is a submersion.*
2. *The set $Z = \{(p,q) \in Q \times Q | \pi(p) = \pi(q)\}$ is a closed submanifold of $Q \times Q$.*

If these two equivalent conditions are satisfied, then the smooth structure on Q/G is unique. The set Q/G endowed with this smooth structure is the *orbit manifold*.

Let $p : G \to H$ be a surjective Lie group homomorphism with kernel K, Q be a left G-manifold, P' be a left H-manifold and $\pi : Q \to P'$ be a p-equivariant surjective submersion, i.e. $\pi(gq) = p(g)\pi(q)$ for all $q \in Q$, $g \in G$. Having p, we can consider every left H-manifold P as a left G-manifold by $gy = p(g)y$ $g \in G$, $y \in P$.

THEOREM 2. *If each $\pi^{-1}(x)$, $x \in P'$ is a K-orbit in Q, then there is a bijection between the G-equivariant maps $\chi : Q \to P$ and the H-equivariant maps $\chi_0 : P' \to P$ given by $\chi = \chi_0 \circ \pi$.*

3. 1st order differential invariants

We now compute the differential invariants on Q^1 with values in left $(L^1_n \times L^1_m)$-manifold. If $q \in Q^1$ and $(A,C) \in L^2_n \times L^2_m$, we denote $\overline{q} = (A^{-1}, C^{-1}) \cdot q$. Let $h_{ij}, g_{\sigma\nu}$, $h_{ij,k}$, $g_{\sigma\nu,\eta}$, $f^\sigma_k, f^\sigma_{ij}$, where $1 \le i \le j \le n$, $1 \le k \le n$, $1 \le \sigma \le \nu \le m$, $1 \le \eta \le m$, be the canonical coordinates on Q^1, and let (a^i_j, a^i_{jk}) (resp. $(c^\sigma_\nu, c^\sigma_{\nu\eta})$) be the canonical coordinates on L^2_n (resp. L^2_m). Let $A^{-1} = (b^i_j, b^i_{jk})$, $C^{-1} = (d^\sigma_\nu, d^\sigma_{\nu\eta})$. If we denote $h_{ij}(\overline{q}) = \overline{h}_{ij}$, $g_{\sigma\nu}(\overline{q}) = \overline{g}_{\sigma\nu}$, $h_{ij,k}(\overline{q}) = \overline{h}_{ij,k}$, $g_{\sigma\nu,\eta}(\overline{q}) = \overline{g}_{\sigma\nu,\eta}$, $f^\sigma_k(\overline{q}) = \overline{f}^\sigma_k$, $f^\sigma_{ij}(\overline{q}) = \overline{f}^\sigma_{ij}$, we get for the action of $L^2_n \times L^2_m$ on Q^1 the expressions

$$
\begin{aligned}
\overline{h}_{ij} &= a^a_i a^b_j h_{ab}, \\
\overline{g}_{\sigma\nu} &= c^\alpha_\sigma c^\beta_\nu g_{\alpha\beta}, \\
\overline{f}^\sigma_k &= d^\sigma_\alpha a^a_k f^\alpha_a, \\
\overline{h}_{ij,k} &= (a^a_{ik} a^b_j + a^a_i a^b_{jk}) h_{ab} + a^a_i a^b_j a^c_k h_{ab,c}, \\
\overline{g}_{\sigma\nu,\eta} &= (c^\alpha_{\sigma\eta} c^\beta_\nu + c^\alpha_\sigma c^\beta_{\eta\nu}) g_{\alpha\beta} + c^\alpha_\sigma c^\beta_\nu c^\gamma_\eta g_{\alpha\beta,\gamma}, \\
\overline{f}^\sigma_{ij} &= d^\sigma_{\nu\kappa} a^k_i a^l_j f^\nu_k f^\kappa_l + d^\sigma_\nu a^k_{ij} f^\nu_k + d^\sigma_\nu a^k_i a^l_j f^\nu_{kl}.
\end{aligned}
$$

We introduce a new global chart on Q^1 adapted to this action, with coordinates $(h_{ij}, g_{\sigma\nu}, f^\sigma_i, \Gamma^i_{jk}, \Delta^\sigma_{\nu\eta}, K^\sigma_{jk})$, with the help of the transformation equations

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1}{2}h^{iq}(h_{qj,k}+h_{qk,j}-h_{jk,q}), & h_{ij,k} &= h_{qi}\Gamma^q_{jk}+h_{qj}\Gamma^q_{ik},\\ \Delta^\sigma_{\nu\eta} &= \frac{1}{2}g^{\sigma\alpha}(g_{\alpha\nu,\eta}+g_{\alpha\eta,\nu}-g_{\nu\eta,\alpha}), & g_{\sigma\nu,\eta} &= g_{\alpha\sigma}\Delta^\alpha_{\nu\eta}+g_{\alpha\nu}\Delta^\alpha_{\sigma\eta},\\ K^\sigma_{jk} &= \Gamma^i_{jk}f^\sigma_i-\Delta^\sigma_{\nu\eta}f^\nu_j f^\eta_k-f^\sigma_{jk}. & f^\sigma_{jk} &= \Gamma^i_{jk}f^\sigma_i-\Delta^\sigma_{\nu\eta}f^\nu_j f^\eta_k-K^\sigma_{jk}.\end{aligned}$$

The coordinates Γ^i_{jk},$\Delta^\sigma_{\nu\eta}$, K^σ_{jk} are symmetric in the subscripts. We put $h^{ij}h_{jk}=\delta^i_k$, $g^{\sigma\nu}g_{\nu\eta}=\delta^\sigma_\eta$. The action of the group $L^2_n\times L^2_m$ on Q^1 is expressed as follows:

$$\begin{aligned}\overline{\Gamma}^i_{jk} &= b^i_a a^b_j a^c_k\Gamma^a_{bc}+b^i_l a^l_{jk},\\ \overline{\Delta}^\sigma_{\nu\eta} &= d^\sigma_\alpha c^\beta_\nu c^\gamma_\eta\Delta^\alpha_{\beta\gamma}+d^\sigma_\alpha c^\alpha_{\nu\eta},\\ \overline{K}^\sigma_{jk} &= d^\sigma_\alpha a^p_j a^l_k K^\alpha_{pl}.\end{aligned}$$

The action of $K^{2,1}_n\times K^{2,1}_m$ on Q^1 is expressed by

$$\begin{aligned}\overline{h}_{ij} &= h_{ij}, & \overline{\Gamma}^i_{jk} &= \Gamma^i_{jk}+a^i_{jk},\\ \overline{g}_{\sigma\nu} &= g_{\sigma\nu}, & \overline{\Delta}^\sigma_{\nu\eta} &= \Delta^\sigma_{\nu\eta}+c^\sigma_{\nu\eta},\\ \overline{f}^\sigma_i &= f^\sigma_i, & \overline{K}^\sigma_{jk} &= K^\sigma_{jk}.\end{aligned} \tag{1}$$

With the help of expressions (1) we can describe the quotient space $Q^1/(K^{2,1}_n\times K^{2,1}_m)$. The following theorem is the direct consequence of Theorem 1.

THEOREM 3. *The set $Q^1/(K^{2,1}_n\times K^{2,1}_m)$ has the orbit manifold structure. In the global coordinates $(h_{ij},g_{\sigma\nu},f^\sigma_i,\Gamma^i_{jk},\Delta^\sigma_{\nu\eta},K^\sigma_{jk})$ on Q^1 and the global coordinates $(h_{ij},g_{\sigma\nu},f^\sigma_i,K^\sigma_{jk})$ on $Q^1/(K^{2,1}_n\times K^{2,1}_m)$, the quotient projection π is expressed by*

$$(h_{ij},g_{\sigma\nu},f^\sigma_i,\Gamma^i_{jk},\Delta^\sigma_{\nu\eta},K^\sigma_{jk})\to(h_{ij},g_{\sigma\nu},f^\sigma_i,K^\sigma_{jk}).$$

.

To study the structure of the orbit manifold $Q^1/(K^{2,1}_n\times K^{2,1}_m)$, consider the manifold

$$Q_0=\mathrm{Met}\mathbf{R}^n\times\mathrm{Met}\mathbf{R}^m\times\mathrm{reg}\,(\mathbf{R}^{n*}\otimes\mathbf{R}^m)\times(\mathbf{R}^{n*}\odot\mathbf{R}^{n*}\otimes\mathbf{R}^m)$$

with the tensor action of the group $L^1_n\times L^1_m$. The set $Q^1/(K^{2,1}_n\times K^{2,1}_m)$ is endowed with the structure of a left $(L^1_n\times L^1_m)$-manifold given by

$$(L^1_n\times L^1_m)\times Q^1/(K^{2,1}_n\times K^{2,1}_m)\ni((j^1_0\alpha,J^1_0\gamma),[q]_K)\to$$
$$\to[(i^{1,2}_n(j^1_0\alpha),i^{1,2}_m(j^1_0\gamma))\cdot q]_K\in Q^1/(K^{2,1}_n\times K^{2,1}_m),$$

where $i^{1,2}_n:L^1_n\to L^2_n$ is canonical inclusion. The following assertion is an immediate consequence of the existence of global coordinates on both $Q^1/(K^{2,1}_n\times K^{2,1}_m)$ and Q_0.

THEOREM 4. *The left* $(L^1_n \times L^1_m)$*-manifold* $Q^1/(K^{2,1}_n \times K^{2,1}_m)$ *is isomorphic with* Q_0.

The following theorem is a consequence of expression (1) and Theorem 2.

THEOREM 5. *Every differential invariant* $\chi : Q^1 \to P$ *has a unique expression in the form* $\chi = \chi_0 \circ \pi$, *where* $\chi_0 : Q^1/(K^{2,1}_n \times K^{2,1}_m) \to P$ *is* $(L^1_n \times L^1_m)$*-equivariant mapping.*

COROLLARY. *Theorem 5 means that the coordinate functions* h_{ij}, $g_{\sigma\nu}$, f^σ_i, K^σ_{ij} *on* $Q^1/(K^{2,1}_n \times K^{2,1}_m)$, *form a* basis *of the first order differential invariants.*

4. Geometrical interpretation of adapted coordinates

In this section we consider an injective immersion $f : X \to Y$. Denote by $\mathcal{V}(X)$ resp. $\mathcal{V}(Y)$ the modules of smooth vector fields on X resp. Y.

Let

$$\nabla^h : \mathcal{V}(X) \times \mathcal{V}(X) \ni (u, v) \to \nabla^h_u v \in \mathcal{V}(X)$$

resp.

$$\nabla^g : \mathcal{V}(Y) \times \mathcal{V}(Y) \ni (\alpha, \beta) \to \nabla^g_\alpha \beta \in \mathcal{V}(Y)$$

be the covariant derivative on the manifold X resp. Y given by the Levi—Civita connection $\Gamma^i_{jk} = (1/2)h^{iq}(h_{qj,k} + h_{qk,j} - h_{jk,q})$ resp. $\Delta^\sigma_{\nu\eta} = (1/2)g^{\sigma\alpha}(g_{\alpha\nu,\eta} + g_{\alpha\eta,\nu} - g_{\nu\eta,\alpha})$.

Recall that any vector field v on X, defines a vector field along f, $f_* v$. Choose a point $x \in X$. There exists a neighbourhood V of $y = f(x)$ and a vector field $\tilde{v}$ on V such that $\tilde{v} = f_* v$ on $V \cap f(X)$. Consider a vector field $\nabla^g_{\tilde{u}} \tilde{v}$ on V and its restriction to $V \cap f(X)$. One can easily verify then this construction gives us a well-defined vector field along f.

To every $x \in X$ we assign a bilinear mapping $K_f(x) : T_x X \times T_x X \to T_{f(x)} Y$ such that

$$K_f(x)(u(x), v(x)) = f_*(\nabla^h_u v(x)) - (\nabla^g_{\tilde{u}} \tilde{v})(f(x)),$$

where $\tilde{u}$ and $\tilde{v}$ are any extensions of $f_* u$ and $f_* v$ on a neighbourhood of $f(x)$. This formula defines a bilinear mapping $K_f : \mathcal{V}(X) \times \mathcal{V}(X) \to \mathcal{V}(Y)$ along f.

Let (U, φ), $\varphi = (x^i)$, resp. (V, ψ), $\psi = (y^\sigma)$, be a chart on X resp. Y. Let $f(u) \subset V$. Denote $u = u^j \partial/\partial x^j$, $v = v^k \partial/\partial x^k$. We express $K_f(x)$ in components. We have for every $x \in U$

$$K_f(x)(u(x), v(x)) = K^\sigma_{jk}(x) u^i(x) v^j(x) (\frac{\partial}{\partial y^\sigma})_{f(x)}.$$

THEOREM 6. *The components of* K_f *are*

$$K^\sigma_{jk} = \Gamma^i_{jk} f^\sigma_i - \Delta^\sigma_{\nu\eta} f^\nu_j f^\eta_k - f^\sigma_{jk}.$$

REMARK: The bilinear field K_f along f can also be interpreted within the theory of connections on vector bundles and their pull-back bundles. (See e.g. [5].)

Recall that a mapping $f : X \to Y$ is said to be *affine* if for every geodesic $\alpha : (-\varepsilon, \varepsilon) \to X$ the curve $f \circ \alpha : (-\varepsilon, \varepsilon) \to Y$ is a geodesic. Using this definition, we get by a direct computation:

THEOREM 7. *The following conditions are equivalent:*

1. $K^\sigma_{jk} = 0$.
2. *The mapping f is affine.*

REMARK. In the special case of Riemannian metric g on Y and induced metric $h = f^* g$ on X, the mapping K_f corresponds to the second fundamental forms (associated with different normal vector fields to $f(x)$).

5. Natural Lagrangian structures

In this section we show that the basis of the first order differential invariants, given by Theorem 5, can be used for a description of all possible, first order generally covariant Lagrangians depending on immersions of manifolds with metric fields. To this purpose we have to extend the standard theory of natural Lagrange structures (see [9]) to fibrations over pairs of manifolds.

In general, one should consider for a Lagrangian a differential invariant $\mathcal{L} : Q \to \tilde{\mathbf{R}}$, where $\tilde{\mathbf{R}}$ is the real line endowed with the action $(A,t) \to |\det A|^{-1} \cdot t$ of $GL_n(\mathbf{R})$. However, if Q includes a *metric field*, then the corresponding metric volume element allows us to consider the functions L instead of $\mathcal{L}$, that is, the *Lagrange functions* instead of *odd base forms* on underlying jet manifolds. For the categorical aspects of the theory of natural Lagrange structures, and for the theory of odd base forms the reader is refered to [9] and [7]. Thus, for our purposes it is sufficient to recall a simplified version of the definition, which can be used on manifolds with metric fields.

We define a *natural Lagrange structure* to be a pair (Q, L), where Q is a left L^r_n-manifold, and $L : Q \to \mathbf{R}$ is an L^r_n-invariant function (i.e., a real-valued differential invariant).

Using this definition, we can paraphrase Theorem 5 as follows:

THEOREM 8. *Every generally covariant 1st order Lagrange function depending on an immersion $f : X \to Y$ and metric fields on X and Y, respectively, is a function of coordinates h_{ij}, $g_{\sigma\nu}$, f^σ_i, K^σ_{ij} only.*

In particular, for the induced metrics we have the following result:

COROLLARY. *Let $f : X \to Y$ be an injective immersion, let g be a Riemannian metric on Y and let $h = f^* g$ be the induced metric on X. Then every 1st order invariant Lagrange function depends on the metric tensor components, the Jacobian matrix (f^σ_i), and the second fundamental form components only.*

In accordance with the results of the classical theory of invariants (see [2], [13]), every algebraic invariant function on spaces endowed with a tensor action can by constructed by operations of the tensor product, alternation and contraction. The typical example of a zeroth order Lagrange function is the energy density, mentioned in the introduction, and used e.g. in the string theory. An example of the 1st order Lagrange function is given by $L_1 = h^{ij} h^{ab} K^\sigma_{ia} K^\nu_{jb} g_{\sigma\nu}$; this Lagrangian usually serves as the geometric part of a higher order Lagrangian in the string theory (see e.g. [3]).

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