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Differential Invariants of Immersions of Manifolds with Metric Fields

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Abstract: The problem of finding all r-th order differential invariants of immersions of manifolds with metric fields, with values in a left $(G^1_m \times G^1_n)$-manifold is formulated. For obtaining basis of higher order differential invariants the orbit reduction method is used.

As a new result it appears that r-th order differential invariants depending on an immersion $f : M \to N$ of smooth manifolds M and N and metric fields on them can be factorized through metrics, curvature tensors and their covariant differentials up to the order $(r-2)$, and covariant differentials of the tangent mapping Tf up to the order r. The concept of a covariant differential of Tf is also introduced in this paper. The obtained results are geometrically interpreted as well.

Keywords: differential invariant, immersion, orbit reduction method, covariant derivative.

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1. Introduction

Many important problems of the theory of invariants have undoubtedly their motivations and applications in physical theories. The problem of describing all higher order differential invariants depending on some fields defined and having their values on smooth manifolds is considered. An meaningfull particular case of this general problem is that of finding higher order differential invariants of an immersion $f : M \to N$ of smooth manifolds M, N endowed with metric fields h and g, respectively.

This can be understood as a generalization of the requirement to describe all zero and first order generally covariant Lagrangians depending on an immersion

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of smooth manifolds with metric fields which are widely used in physical theories, especially in the string theory (see e.g. [5]).

Recall that typical example of such a Lagrangian is the well-known energy density, the zeroth-order differential invariant $e(f) = 1/2(f^*g)_{ij}h^{ij}$ defining the energy functional

$$E(f) = \int_M e(f)(x)\sqrt{\det(h_{ij})(x)}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$$

with harmonic mappings as critical points (see [8]).

In this paper the basis of r-th order differential invariants of immersions of smooth manifolds with metric fields is found. With the use of this basis, all differential invariants can be constructed. The complete solution of the problem of construction of higher order differential invariants is presented in [18], with the use of general methods given in [4] and [20]. As an effective tool for obtaining the basis, the orbit reduction method is used. This method was first used in [12] and then developed in [10]. Its practical usefulness in invariant theory was verified in [19] for the case of invariants of metric tensor field on a smooth manifold. It was also used in [17] for solving the problem of first-order differential invariants of immersions of smooth manifolds with metric fields.

Main theorems for the practical use of the orbit reduction method are reproduced in section 2 of the just presented paper. Section 3 summarizes the well-known definitions and concepts of the theory of invariants (see [1], [3], [9]) which we need for our considerations, reformulated with the use of the orbit reduction method. These results concern the case of a single manifold with a metric field. Remaining sections present our own results for the case of an immersion of two manifolds with metric fields.

Section 4 extends methods used in [17] for finding first-order invariants to the case of arbitrary order. The basis of higher order invariants enables us to construct all invariants of the corresponding order, depending on an immersion $f : M \to N$ of smooth manifolds M, N and their metric fields h, g, respectively.

As a main result we show that such r-th order differential invariants can be factorized through both metrics and corresponding curvature tensors, as well as their covariant differentials up to the order $r - 2$ and covariant differentials of the tangent mapping Tf induced by f up to the order r. The new concept of the r-th order covariant differential of Tf introduced in this paper, gives an appropriate geometrical interpretation to obtained results in section 5. Finally, some practical consequences of our results that can be used for applications in physical theories are formulated in section 6.

2. Orbit reduction method

In this section we recall two fundamental theorems which the orbit reduction method is based on. Such formulations of these theorems are presented that will be appropriate for their direct application in finding differential invariants.

The first of promised theorems concerns the smooth structure of an orbit manifold. Its proof can be found e.g. in the classical book [2].

Theorem 1. *Let G be a Lie group. Let Q be a connected left G-manifold, $\pi : Q \to Q/G$ the corresponding quotient projection. The following two conditions are equivalent:*

1. *There exists a smooth structure on Q/G such that π is a submersion.*
2. *The set $Z = \{(p,q) \in Q \times Q | \pi(p) = \pi(q)\}$ is a closed submanifold of $Q \times Q$.*

If there exists a smooth structure on Q/G such that π is a submersion, then it is unique. As a consequence it can be stated:

Definition 1. *The set Q/G endowed by the smooth structure from Theorem 1 (if exists) is called the* **orbit manifold**.

Let G, H be Lie groups and let $p : G \to H$ be a surjective Lie group homomorphism. Let Q be a left G-manifold and P a left H-manifold. Then we can define a left action of the group G on the manifold P as follows

$$G \times P \ni (g,y) \to gy = p(g)y \in P$$

for every $g \in G$ and every $y \in P$. Thus, P becomes a left G-manifold. The following holds:

Theorem 2. *Let $p : G \to H$ be a surjective Lie group homomorphism with the kernel K. Let Q be a left G-manifold and P a left H-manifold. Let P' be a left H-manifold such that*

1. *there exists a p-equivariant surjective submersion $\pi : Q \to P'$, i.e. $\pi(gq) = p(g)\pi(q)$ for all $q \in Q$, $g \in G$,*
2. *for every point $x \in P'$ the set $\pi^{-1}(x)$ is a K-orbit in Q.*

Then there exists a bijection between the smooth G-equivariant maps $\chi : Q \to P$ and the smooth H-equivariant maps $\chi_0 : P' \to P$ given by $\chi = \chi_0 \circ \pi$.

The proof of this theorem is immediate and it can be found e.g. in [9].

3. Differential invariants of a metric tensor field

This section summarizes some familiar concepts and results of the theory of differential invariants which we shall need for our later considerations. Main classical theorems on differential invariants were obtained in earlier works (see e. g. [1], [3]), where they are related to the theory of partial differential equations (heat equation), index theory, etc.). However, the reformulation of these results with the use of the concept of jets and the orbit reduction method is very effective for solving our problem of differential invariants of an immersion of manifolds with metric fields. Thus, for our purposes we use the works [9], [10], where the theory of invariants based on the orbit reduction method is developed in general, and the paper [19] containing some specific results.

Let G^r_m be the r-th differential group of the Euclidean space $\mathbf{R}^m$. (Recall that this is the set of all regular r-jets with the source and target in $0 \in \mathbf{R}^m$, endowed with the smooth structure given by the canonical global chart and with the Lie group structure given by composition of jets.) Let Q be a left G^r_m-manifold and P a left G^s_m-manifold, for $r \geq s$. Denote by $\pi^{r,s}_m : G^r_m \to G^s_m$ the canonical projection.

Definition 2. *A mapping $\chi : Q \to P$ is called a* **differential invariant** *if it satisfies the condition*

$$\chi(j_0^r\alpha \cdot q) = \pi_m^{r,s}(j_0^r\alpha) \cdot \chi(q)$$

for all $q \in Q$ and all $j_0^r\alpha \in G_m^r$.

Let us discuss a typical example of the differential invariants that will be meaningful in our considerations — differential invariants of metric fields.

Let $\mathbf{R}^{m*}$ be the dual space to $\mathbf{R}^m$. Denote by $\mathrm{Met}\mathbf{R}^m \subset \mathbf{R}^{m*} \odot \mathbf{R}^{m*}$ the set of all regular metric tensors and $T_m^r\mathrm{Met}\mathbf{R}^m$ the manifold of r-jets with the source $0 \in \mathbf{R}^m$ and target in $\mathrm{Met}\mathbf{R}^m$. We define a left action of the group G_m^{r+1} on $T_m^r\mathrm{Met}\mathbf{R}^m$ by

$$G_m^{r+1} \times T_m^r\mathrm{Met}\mathbf{R}^m \ni (j^{r+1}\alpha, j_0^r h) \to j_0^r(j_0^1\alpha \cdot h) \in T_m^r\mathrm{Met}\mathbf{R}^m.$$

In this relation the mapping $j_0^1\alpha \cdot h : \mathbf{R}^m \to \mathrm{Met}\mathbf{R}^m$ is defined by

$$j_0^1\alpha \cdot h(x) = j_0^1(t_x\alpha t_{-\alpha^{-1}(x)}) \cdot h(\alpha^{-1}(x)) \tag{1}$$

with the multiplication on the righthand side given by the tensor action of the group $GL(n,\mathbf{R})$, and $t_x : \mathbf{R}^m \ni y \to t_x(y) = y - x \in \mathbf{R}^m$ is the translation.

Let M be a smooth manifold of dimension m. We can interprete the manifold $T_m^r\mathrm{Met}\mathbf{R}^m$ as a type fiber of the fiber bundle $P^{r+1}M \times_{G_m^{r+1}} T_m^r\mathrm{Met}\mathbf{R}^m$, associated with the principal G_m^{r+1}-bundle $P^{r+1}M$ of frames.

Definition 3. *Let P be a left G_m^1-manifold. The differential invariant $\chi : T_m^r\mathrm{Met}\mathbf{R}^m \to P$ is called* **r-th order differential invariant of the metric field***.*

Denote by $K_m^{r+1,1}$ the kernel of the canonical group morphism $\pi_m^{r+1,1} : G_m^{r+1} \to G_m^1$. $K_m^{r+1,1}$ is a nilpotent normal subgroup of G_m^{r+1}. The quotient space $T_m^r\mathrm{Met}\mathbf{R}^m/K_m^{r+1,1}$ has, by Theorem 1, the uniquely defined smooth structure of the orbit manifold, moreover it has the structure of a left G_m^1-manifold, see [19]. Due to the Theorem 2, every differential invariant $\chi : T_m^r\mathrm{Met}\mathbf{R}^m \to P$ factorizes through an G_m^1-equivariant mapping $\chi_0 : T_m^r\mathrm{Met}\mathbf{R}^m/K_m^{r+1,1} \to P$. This means, that any set of coordinates on the mentioned quotient space can be viewed as a basis of differential invariants. Such basis is explicitely described by the below presented results, proved in [19].

Let functions (h_{ij}), $1 \le i,j \le m$, be the components of a metric $h \in \mathrm{Met}\mathbf{R}^m$, functions Γ_{jk}^i the components of the corresponding Levi-Civita connection and R_{ijkl} the components of the curvature tensor R:

$$\Gamma_{jk}^i = \frac{1}{2}h^{iq}(h_{qj,k} + h_{qk,j} - h_{jk,q}), \tag{2}$$

$$R_{ijkl} = \frac{1}{2}(h_{il,jk} + h_{jk,il} - h_{ik,jl} - h_{jl,ik}) + h_{bc}(\Gamma_{jk}^b\Gamma_{il}^c - \Gamma_{jl}^b\Gamma^c ik), \tag{3}$$

moreover for $1 \le s \le r-2$ consider the covariant derivatives $(R_{ijkl;m_1;\ldots;m_s})$ of the curvature tensor (3) and for $1 \le t \le r-1$ consider functions $(\Gamma_{jk,l_1\ldots l_t}^i)$ defined as t-th derivative of (2). As it is proved in [19] the every independent subsystem of the system of functions

$$(h_{ij}, \Gamma_{jk,l_1\ldots l_t}^i, R_{ijkl;m_1;\ldots;m_s}),$$

where $1 \leq i,j,k,l \leq m$, $0 \leq s \leq r-2$, $1 \leq t \leq r-1$ defines a global chart on $T^r_m\mathrm{Met}\mathbf{R}^m$ adapted to the action of the group G^{r+1}_m. Any independent subsystem of the system of functions

$$(h_{ij}, R_{ijkl;m_1;\dots;m_s})$$

defines a global chart on $T^r_m\mathrm{Met}\mathbf{R}^m/K^{r+1,1}_m$ that gives the orbit manifold structure on this set. Following theorem holds.

Theorem 3. *Let P is a left G^1_m-manifold. Every r-th order differential invariant $\chi : T^r_m\mathrm{Met}\mathbf{R}^m \to P$ is a function of components of the metric tensor (h_{ij}), components of the curvature tensor and its covariant differentials $(R_{ijkl;m_1;\dots;m_s})$, $0 \leq s \leq r-2$ only.*

4. Differential invariants of immersions of smooth manifolds with metric fields

Let M and N be smooth manifolds, $\dim M = m$, $\dim N = n$, $m \leq n$, endowed with metric fields h and g, respectively. Let $f : M \to N$ be an immersion. For $r \geq 0$ denote

$$Q^r = T^r_m\mathrm{Met}\mathbf{R}^m \times T^r_n\mathrm{Met}\mathbf{R}^n \times \mathrm{Imm}J^{r+1}_{(0,0)}(\mathbf{R}^m, \mathbf{R}^n)$$

where $\mathrm{Imm}J^{r+1}_{(0,0)}(\mathbf{R}^m, \mathbf{R}^n)$ is the manifold of regular $(r+1)$-jets with the source $0 \in \mathbf{R}^m$ and target $0 \in \mathbf{R}^n$. The set Q^r is endowed with a natural left action of the Lie group $G^{r+1}_m \times G^{r+1}_n$ given by

$$(G^{r+1}_m \times G^{r+1}_n) \times Q^r \ni ((j^{r+1}_0\alpha, j^{r+1}_0\gamma), (j^r_0 h, j^r_0 g, j^{r+1}_0 f)) \to$$

$$\to (j^r_0(j^1_0\alpha \cdot h), j^r_0(j^1_0\gamma \cdot g), j^{r+1}_0(\gamma \circ f \circ \alpha^{-1})) \in Q^r \tag{4}$$

where $j^1_\alpha \cdot h$ is given by (1) and $j^1_0\gamma \cdot g$ is defined by the quite analogous way.

Remark 1. In fact, we consider the fiber bundle $(P^{r+1}M \times P^{r+1}N) \times_{G^{r+1}_m \times G^{r+1}_n} Q^r$ with the type fiber Q^r, associated with the principal $(G^{r+1}_m \times G^{r+1}_n)$-bundle $P^{r+1}M \times P^{r+1}N$ of frames. The formula (4) represents the transformations, induced on two metric fields and their derivatives as well as derivatives of an immersion by the coordinate trasformations (at a point of $\mathbf{R}^m \times \mathbf{R}^n$), see e.g. [10].

Definition 4. *Let P be a left $(G^1_m \times G^1_n)$-manifold, A smooth mapping $\chi : Q^r \to P$ such that*

$$\chi((j^{r+1}_0\alpha, j^{r+1}_0\gamma) \cdot q) = (\pi^{r+1,1}_m(j^{r+1}_0\alpha), \pi^{r+1,1}_n(j^{r+1}_0\gamma)) \cdot \chi(q)$$

for all $q \in Q^r$, $j^{r+1}_0\alpha \in G^{r+1}_m$, $j^{r+1}_0\gamma \in G^{r+1}_n$, is called r**-th order differential invariant of an immersion of manifolds with metric fields**.

In agreement with section 3 denote $K_m^{r+1,1}$ resp. $K_n^{r+1,1}$ the kernel of $\pi_m^{r+1,1}$ resp. $\pi_n^{r+1,1}$. $K_m^{r+1,1} \times K_n^{r+1,1}$ is a normal subgroup of $G_m^{r+1} \times G_n^{r+1}$. In the following we give the description of the quotient space $Q^r/(K_m^{r+1,1} \times K_n^{r+1,1})$ for arbitrary r. This quotient has the structure of a left $(G_m^1 \times G_n^1)$-manifold.

Denote by

$$h_{ij}, h_{ij,k_1}, \dots, h_{ij,k_1\dots k_r}, g_{\sigma\nu}, g_{\sigma\nu,\eta_1}, \dots, g_{\sigma\nu,\eta_1,\dots,\eta_r}, f^\sigma_{k_1}, \dots, f^\sigma_{k_1,\dots,k_{r+1}},$$

the canonical coordinates on Q^r, latin indices running from 1 to m, greek ones from 1 to n (this we assume further). Let $(a^i_{j_1}, \dots, a^i_{j_1\dots j_{r+1}})$ and $(c^\sigma_{\nu_1}, \dots, c^\sigma_{\nu_1\dots\nu_{r+1}})$ be canonical coordinates on G_m^{r+1} and G_n^{r+1}, respectively, and denote $b^i_{j_1\dots j_k}(A) = a^i_{j_1\dots j_k}(A^{-1})$ for $A \in G_m^{r+1}$ and $d^\sigma_{\nu_1\dots\nu_k}(C) = c^\sigma_{\nu_1\dots\nu_k}(C^{-1})$ for $C \in G_n^{r+1}$. For the action of $G_m^{r+1} \times G_n^{r+1}$ in canonical coordinates the following recurent formula holds. Let for $1 < s \le r+1$

$$\overline{f}^\sigma_{k_1\dots k_{s-1}} = p(a^i_{k_1}, \dots, a^i_{k_1\dots k_{s-1}}, d^\sigma_{\nu_1}, \dots, d^\sigma_{\nu_1\dots\nu_{s-1}}, f^\nu_{i_1}, \dots, f^\nu_{i_1\dots i_{s-1}}).$$

Then

$$\overline{f}^\sigma_{k_1\dots k_s} = \sum_{t=1}^{s-1} \left(\frac{\partial p}{\partial a^i_{j_1\dots j_t}} a^i_{j_1\dots j_t k_s} + \frac{\partial p}{\partial d^\alpha_{\nu_1\dots\nu_t}} d^\alpha_{\nu_1\dots\nu_t\beta} f^\beta_c a^c_{k_s} + \frac{\partial p}{\partial f^\alpha_{i_1\dots i_t}} f^\alpha_{i_1\dots i_t c} a^c_{k_s} \right).$$

The formulas for canonical coordinates on $T^r_m\mathrm{Met}\mathbf{R}^m$ resp. $T^r_n\mathrm{Met}\mathbf{R}^n$ are well-known and the reader can find them in [19], [18].

We introduce new coordinates (adapted coordinates) on Q^r: $(h_{ij}, \Gamma^i_{jk,l_1\dots l_t}, R_{ijkl;m_1;\dots;m_s}, g_{\sigma\nu}, \Delta^\sigma_{\nu\eta\gamma_1\dots\gamma_t}, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_s}, K^\sigma_{i_1\dots i_p})$, $0 \le s \le r-2$, $0 \le t \le r-1$, $1 \le p \le r+1$ where

$$(h_{ij}, \Gamma^i_{jk,l_1\dots l_t}, R_{ijkl;m_1;\dots;m_s})$$

are coordinates on $T^r_m\mathrm{Met}\mathbf{R}^m$ adapted to the action of G_m^{r+1}, resp.

$$(g_{\sigma\nu}, \Delta^\sigma_{\nu\eta,\gamma_1\dots\gamma_t}, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_s})$$

are coordinates on $T^r_n\mathrm{Met}\mathbf{R}^n$ adapted to the action of G_n^{r+1}, given in section 3. Moreover, $K^\sigma_{i_1\dots i_p}$ are given by transformation equations

$$\begin{aligned} K^\sigma_{i_1} &= f^\sigma_{i_1}, \\ K^\sigma_{i_1 i_2} &= K^\sigma_{i_1,i_2} + \Gamma^\sigma_{\alpha\beta} K^\alpha_{i_2} K^\beta_{i_1} - \Gamma^a_{i_2 i_1} K^\sigma_a, \\ &\vdots \\ K^\sigma_{i_1\dots i_{r+1}} &= K^\sigma_{i_1\dots i_r,i_{r+1}} + \Gamma^\sigma_{\alpha\beta} K^\alpha_{i_{r+1}} K^\beta_{i_1\dots i_r} - \\ &\quad -\Gamma^a_{i_{r+1}i_1} K^\sigma_{a i_2\dots i_r} - \Gamma^a_{i_{r+1}i_2} K^\sigma_{a i_1 i_3\dots i_r} - \\ &\quad - \dots - \Gamma^a_{i_{r+1}i_r} K^\sigma_{a i_1\dots i_{r-1}}. \end{aligned} \tag{5}$$

Let I_s be multiindex of length z, $z \in \{2, \dots, r+1\}$. We can write (5) in a form

$$K^\sigma_{I_{z-1}i_z} = [K^\sigma_{I_{z-1},i_z} + \Gamma^\sigma_{\alpha\beta} K^\alpha_{i_z} K^\beta_{I_{z-1}} - \sum_{i \in I_{z-1}} \Gamma^a_{i_z i} K^\sigma_{a(I_{z-1}\setminus\{i\})}].$$

We should specify notation K_{I_{z-1},i_z}. Let $K = K(h_{ij,I_t}, g_{\sigma\nu,\Upsilon_t}, f^\sigma_{I_{t+1}})$, where I_t, Υ_t are multiindices of length t, $t \in \{0, \dots, r-1\}$. By $K_{,u}$ we denote the function given by

$$K_{,u} = \sum_{t=0}^{r-1} \left(\frac{\partial K}{\partial h_{ij,I_t}} h_{ij,I_t u} + \frac{\partial K}{\partial g_{\sigma\nu,\Upsilon_t}} g_{\sigma\nu,\Upsilon_t \eta} f^\eta_u + \frac{\partial K}{\partial f^\sigma_{I_{t+1}}} f^\sigma_{I_{t+1} u} \right).$$

For the action of $G^{r+1}_m \times G^{r+1}_n$ resp. $K^{r+1,1}_m \times K^{r+1,1}_n$ and $p \in \{1, \dots, r+1\}$ we get

$$\overline{K}^\sigma_{i_1 \dots i_p} = a^{j_1}_{i_1} \dots a^{j_p}_{i_p} d^\sigma_\nu K^\nu_{j_1 \dots j_p}$$

resp.

$$\overline{K}^\sigma_{i_1 \dots i_p} = K^\sigma_{i_1 \dots i_p}. \tag{6}$$

As a consequence of Theorem 3, equations (6) and Theorem 1 we get the following theorem.

Theorem 4. *The set $Q^r/(K^{r+1,1}_m \times K^{r+1,1}_n)$ has the orbit manifold structure, given by global coordinate chart with coordinates $(h_{ij}, R_{ijkl}, R_{ijkl;m_1}, \dots, R_{ijkl;m_1;\dots;m_{r-2}}, g_{\sigma\nu}, \varOmega_{\sigma\nu\eta\omega}, \varOmega_{\sigma\nu\eta\omega;\gamma_1}, \dots, \varOmega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_{r-2}}, K^\sigma_{i_1}, \dots, K^\sigma_{i_1 \dots i_{r+1}})$.*

The coordinates $(h_{ij}, R_{ijkl}, R_{ijkl;m_1}, \dots, R_{ijkl;m_1;\dots;m_{r-2}}, g_{\sigma\nu}, \varOmega_{\sigma\nu\eta\omega}, \varOmega_{\sigma\nu\eta\omega;\gamma_1}, \dots, \varOmega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_{r-2}}, K^\sigma_{i_1}, \dots, K^\sigma_{i_1 \dots i_{r+1}})$ form a basis of r-th order differential invariants.

5. Geometrical interpretation

As one of our new results we give the geometrical interpretation of adapted coordinates as a covariant differentials of some objects, defined as sections of natural bundles. For this purpose we use some well-known concepts given in [16], [6], [7].

Definition 5. *By a* **linear connection on a vector bundle** (E, p, M) *we mean a linear section*

$$\varGamma : E \to J^1 E.$$

Proposition 1. *Let $\varGamma$ be a linear connection on E. Then, there is a unique linear connection $\varGamma^* : E^* \to J^1 E^*$ on the dual vector bundle $E^* \to M$ such that the following diagram commutes*

$$\begin{array}{ccc} E \times E^* & \xrightarrow{\;<\ |\ >\;} & M \times \mathbf{R} \\ \Big\downarrow{\scriptstyle \varGamma \times \varGamma^*} & & \Big\downarrow{\scriptstyle 0 \times \mathrm{id}_{\mathbf{R}}} \\ J^1 E \times J^1 E^* & \xrightarrow{\;J^1 <\ |\ >\;} & T^* M \times \mathbf{R} \end{array}$$

where by 0 *we denote the zero section. Note that* $J^1E \times J^1E^*$ *is isomorphic with* $J^1(E \times E^*)$.

Definition 6. *The connection* Γ^* *is said to be the* **dual connection to** Γ.

Proposition 2. *Let* Γ *resp.* Γ' *be a linear connection on vector bundle* E *resp.* E' *over* M. *Then, there is a unique linear connection* $\Gamma \otimes \Gamma' : E \otimes_M E' \to J^1(E \otimes_M E')$ *such that the following diagram commutes*

$$\begin{tikzcd} E \times E' \arrow[r, "\otimes"] \arrow[d, "\Gamma \times \Gamma'"'] & E \otimes_M E' \arrow[d, "\Gamma \otimes \Gamma'"] \\ J^1E \times_M J^1E' \arrow[r, "J^1\otimes"] & J^1(E \otimes_M E') \end{tikzcd}$$

Definition 7. *The connection* $\Gamma \otimes \Gamma'$ *is said to be the* **tensor product connection of** Γ **and** Γ'.

Proposition 3. *Considering the contact mapping* $\mathrm{c} : J^1E \hookrightarrow T^*M \otimes_M TE$ *a linear connection* Γ *can be regarded as a* TE*-valued 1-form*

$$\Gamma : E \to T^*M \otimes TE,$$

projecting on the identity of TM.

Let (E, p, N) be a vector bundle and let $f : M \to N$ be smooth. Consider (see [9]) the pullback vector bundle (f^*E, f^*p, M) together with the vector bundle homomorphism p^*f, which is moreover fiberwise difeomorphism and

$$\begin{tikzcd} f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Let $\Delta : E \to T^*N \otimes TE$ be a connection. Consider mapping $f^*\Delta : f^*E \to T^*M \otimes_M Tf^*E$ given by

$$f^*\Delta(u)(\xi) = (T_u p^*f)^{-1} \circ \Delta(p^*f(u)) \circ (T_{f^*p(u)} f \cdot \xi), \tag{7}$$

where $u \in f^*E$, $\xi \in T_{f^*p(u)}M$.

Proposition 4. *The mapping* $f^*\Delta$ *is a connenction on the vector bundle* f^*E *over* M.

Definition 8. *The connection* $f^*\Delta$ *on* f^*E *given by (7) is called the* **pullback connection of** Δ.

We show some coordinate expressions of connections, which we will need later, general formulas can be found in [16] or [6] and [7]. Let M and N be manifolds of corresponding dimensions m and n, $f : M \to N$ be smooth, let $\Gamma : TM \to J^1TM$ resp. $\Delta : TN \to J^1TN$ be a connection on TM resp. TN. Consider local charts $\varphi = (x^i)$, $i = 1, \dots, m$, resp. $\psi = (y^\sigma)$, $\sigma = 1, \dots, n$, on M resp. N. We denote $(x^i, \dot{x}^i)$ and $(x^i, \dot{x}_i)$ resp. $(y^\sigma, \dot{y}^\sigma)$ and $(y^\sigma, \dot{y}_\sigma)$ the induced coordinate chart on TM and T^*M resp. TN and T^*N, the induced local basis of sections of TM resp. T^*M is denoted by (∂_i) resp. (d^i), the induced local basis of sections of TN resp. T^*N is denoted by (∂_σ) resp. (d^σ). Derivatives of $\psi \circ f \circ \varphi^{-1}$ we denote by $f^\sigma_{i_1 \dots i_k}$, induced coordinate chart on $\otimes^i T^*M \otimes f^*TN$ we denote by $(x^j, w^\alpha_{j_1 \dots j_i})$.

Proposition 5. *The coordinate expressions of connections Δ, $f^*\Delta$ Γ, Γ^*, $\otimes^i\Gamma^* \otimes f^*\Delta$ is of the type*

$$\begin{aligned}
\Delta &= \mathrm{d}^\sigma \otimes (\partial_\sigma + \Delta^\alpha_{\beta\sigma}\dot{y}^\beta\dot{\partial}_\alpha), \\
\Gamma &= \mathrm{d}^i \otimes (\partial_i + \Gamma^j_{ki}\dot{x}^k\dot{\partial}_j), \\
f^*\Delta &= \mathrm{d}^\mathrm{i} \otimes (\partial_i + \Delta^\alpha_{\sigma\beta} f^\sigma_i \dot{y}^\beta \dot{\partial}_\alpha, \\
\Gamma^* &= \mathrm{d}^i \otimes (\partial_i - \Gamma^j_{ik}\dot{x}_j\dot{\partial}^i), \\
\otimes^i\Gamma^* \otimes f^*\Delta &= \mathrm{d}^\mathrm{j}(\partial_j + \Delta^\alpha_{\sigma\beta} f^\sigma_j w^\beta_{k_1\dots k_i} - \Gamma^l_{jk_1} w^\alpha_{lk_2\dots k_i} - \dots - \Gamma^l_{jk_i} w^\alpha_{k_1\dots k_{i-1}l})\partial^{k_1\dots k_i}_\alpha.
\end{aligned}$$

Now we are prepared to introduce new concepts useful for our geometrical considerations. Let $E^i = \otimes^i T^*M \otimes_M f^*TN$ be a vector bundle over M and consider connection $\otimes^i\Gamma^* \otimes f^*\Delta$ on E. Let $s : M \to E^i$ be a section.

Definition 9. *By the* **covariant differential of** s **along** η **with respect to the connections** Γ, Δ **and a smooth mapping** $f : M \to N$, *we mean a section of $\otimes^{i+1}T^*M \otimes f^*TN = E^{i+1}$ given by*

$$\nabla^{(\otimes^i\Gamma^*\otimes f^*\Delta)} s = \mathrm{p}_2 \circ [\mathrm{c} \circ j^1 s - (\otimes^i\Gamma^* \otimes f^*\Delta) \circ s],$$

where $\mathrm{p}_2 : (T^*M \otimes TE^i) \approx (T^*M \otimes (E^i \oplus E^i)) \to T^*M \otimes E^i)$ is the projection on the second summand and $\mathrm{c} : J^1E \hookrightarrow T^*M \otimes_M TE$ is the contact morphism.

Covariant differential may be viewed as a mapping $C^\infty E^i \to C^\infty E^{i+1}$.

Consider the tangent mapping $Tf : TM \to f^*TN$ as an element of $C^\infty E^1$.

Definition 10. *By a* **covariant differential of order** r **of the tangent mapping** Tf *we mean a mapping $[\nabla^{(\Gamma,\Delta,f)}]^r : C^\infty E^1 \to C^\infty E^{r+1}$ given by*

$$[\nabla^{(\Gamma,\Delta,f)}]^r = \overbrace{\nabla^{(\otimes^r\Gamma^*\otimes f^*\Delta)} \circ \nabla^{(\otimes r-1\Gamma^*\otimes f^*\Delta)} \dots \circ \nabla^{(\Gamma^*\otimes f^*\Delta)}}^{r}.$$

Theorem 5. *In coordinates, covariant differential $[\nabla^{(\Gamma,\Delta,f)}]^k Tf$ has the following form:*

$$[\nabla^{(\Gamma,\Delta,f)}]^k Tf = K^\sigma_{i_1\dots i_{k+1}} \partial_\sigma \otimes \mathrm{d}^{\mathrm{i}_1} \otimes \mathrm{d}^{i_{k+1}},$$

where coefficients $K^\sigma_{i_1\dots i_{k+1}}$ are given by recurent formulas (5).

Note to the 1st order invariants

Invariants of the 1st order are meaningful particularly for physical applications. To study the structure of the orbit manifold $Q^1/(K^{2,1}_m \times K^{2,1}_n)$, consider the manifold

$$Q_0 = \mathrm{Met}\mathbf{R}^m \times \mathrm{Met}\mathbf{R}^n \times \mathrm{reg}\,(\mathbf{R}^{m*} \otimes \mathbf{R}^n) \times (\mathbf{R}^{m*} \odot \mathbf{R}^{m*} \otimes \mathbf{R}^n)$$

with the tensor action of the group $G^1_m \times G^1_n$. The set $Q^1/(K^{2,1}_m \times K^{2,1}_n)$ is endowed with the structure of a left $(G^1_m \times G^1_n)$-manifold given by

$$(G^1_m \times G^1_n) \times Q^1/(K^{2,1}_m \times K^{2,1}_n) \ni ((j^1_0\alpha, J^1_0\gamma), [q]_K) \to$$
$$\to [(i^{1,2}_n(j^1_0\alpha), i^{1,2}_m(j^1_0\gamma)) \cdot q]_K \in Q^1/(K^{2,1}_n \times K^{2,1}_m),$$

where $i^{1,2}_m : G^1_m \to G^2_m$ is the canonical inclusion. The following assertion is an immediate consequence of the existence of global coordinates on both $Q^1/(K^{2,1}_m \times K^{2,1}_n)$ and Q_0.

Theorem 6. *The left $(G^1_m \times G^1_n)$-manifold $Q^1/(K^{2,1}_m \times K^{2,1}_n)$ is isomorphic with Q_0.*

Recall following definition.

Definition 11. *A mapping $f : M \to N$ is said to be* **affine** *if for every geodesic $\alpha : (-\varepsilon, \varepsilon) \to M$ the curve $f \circ \alpha : (-\varepsilon, \varepsilon) \to N$ is a geodesic.*

Using this we get by a direct computation:

Theorem 7. *The following conditions are equivalent:*

1. $\nabla^{(\Gamma^* \otimes f^* \Delta)} Tf = 0$.
2. *The mapping f is affine.*

Remark 2. In the special case of Riemannian metric g on N and the induced metric $h = f^* g$ on M, $\nabla^{(\Gamma^* \otimes f^* \Delta)} Tf = \nabla Tf$ corresponds to the second fundamental form. (See [8], [14].)

6. Physical application

In this section we show how the basis of differential invariants can be used for a construction of invariant Lagrangians. We give here only simplified definitions and some theorems useful for practical applications. For the cathegorical aspects of the theory of natural Lagrange structures and for the theory of odd base forms the reader is refered to [13] and [11].

Let $f : M \to N$ be again an immersion of manifolds with metric fields h and g, respectively.

Definition 12. *An m-form on M*

$$\Lambda = L(h_{ij}(x), \ldots, h_{ij,k_1\ldots k_r}(x), g_{\sigma\nu}(f(x)), \ldots, g_{\sigma\nu,\eta_1\ldots\eta_r}(f(x)),$$
$$f^\sigma_i(x), \ldots, f^\sigma_{i_1\ldots i_{r+1}}(x))\sqrt{|\det(h_{ij})|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$$

is called r**-th order invariant Lagrangian of the immersion** f*, if $L : Q^r \to \mathbf{R}$ is a $(G^{r+1}_m \times G^{r+1}_n)$-invariant function.*

In the sense of this definition the form $\Lambda = 1/2(f^*g)_{ij}h^{ij}\sqrt{|\det(h_{ij})|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$ is the invariant Lagrangian of the immersion f. Invariant Lagrangians of immersion are usually used in the string theory (see e.g. the 1st order Lagrangian given by $L = g_{\alpha\beta}K^{\alpha}_{ia}K^{\beta}_{jb}h^{ij}h^{ab}$ introduced in [5] as a geometrical part of physical Lagrangian.)

As a consequence of Theorem 4 we can state following corollaries for invariant Lagrangians of immersion

Corollary 1. *Any invariant r-th order Lagrangian of immersion of manifolds with metric fields can be written in a form*

$$\Lambda = L(h_{ij}, R_{ijkl}, R_{ijkl;m_1}, \ldots, R_{ijkl;m_1;\ldots;m_{r-2}}, g_{\sigma\nu}, \Omega_{\sigma\nu\eta\omega}, \Omega_{\sigma\nu\eta\omega;\gamma_1}, \ldots,$$

$$\Omega_{\sigma\nu\eta\omega;\gamma_1;\ldots;\gamma_{r-2}}, K^{\sigma}_{i_1}, \ldots, K^{\sigma}_{i_1\ldots i_{r+1}})\sqrt{|\det(h_{ij})|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m,$$

where $L : Q^r/(K^{r+1,1}_m \times K^{r+1,1}_n) \to \mathbf{R}$ is a $(G^1_m \times G^1_n)$-invariant function.

Remark 3. Thus the problem of finding all r-th order invariant Lagrangians of immersion reduces to the problem of constructing all $(G^1_m \times G^1_n)$-invariant functions L.

The physical interpretation of this result (using Theorem 3, Theorem 5 and remark 2) is the following:

Corollary 2. *Any invariant r-th order Lagrangian of immersion of manifolds with metric fields depends on metric fields, curvature tensors and their covariant differentials up to the order $r-2$ and covariant differentials of the tangent mapping up to the order r.*

Corollary 3. *Let $f : M \to N$ be an injective immersion, g be a Riemannian metric on N and let $h = f^*g$ be the induced metric on M. Then every 1st order invariant Lagrangian of f depends on the metric tensor components, the Jacobian matrix and the second fundamental form components only.*

It is evident that the quotient space $Q^r/(K^{r+1,1}_m \times K^{r+1,1}_n)$ is isomorphic with some Euclidean space with the tensorial action of the group $G^1_m \times G^1_n$. Then, using the methods of classical theory of invariants (see [4], [20], [15]) it can be proved (see [18]):

Theorem 8. *Any algebraic $(G^1_m \times G^1_n)$-invariant function $L : Q^r/(K^{r+1,1}_m \times K^{r+1,1}_n) \to \mathbf{R}$ can be constructed from the basis of differential invariant by operations of the tensor product, alternation and contraction.*

So, the basis of differential invariants given by Theorem 4 together with Theorem 8 gives a complete solution of construction of all algebraic invariant r-th order Lagrangians of immersions of manifolds with metric fields.

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