Connection induced geometrical concepts*

Pavla Musilová and Jana Musilová

February 21, 2005

Abstract

Geometrical concepts induced by a smooth mappings $f : M \to N$ of manifolds with linear connections are introduced, especially the (higher order) covariant differentials of the mapping tangent to f and the curvature of a corresponding tensor product connection. As an useful and physically meaningful consequence a basis of differential invariants for natural operators of such smooth mappings is obtained for metric connections. A relation to geometry of Riemannian manifolds is discused.

1 Introduction

In physical theories the theory of invariants is frequently used. One of specific problems, the solution of which can be applied in physical theories, is that of differential invariants of smooth mappings of manifolds with metric fields. This problem was initially solved for first order invariants in [14], and then for general order invariants in [13]. Later it was shown (see [15]) that such invariants can be geometrically interpreted by the concept of natural operators. In the presented paper we show that the approach considered in [15] can be understood as a consequence of a more general situation in which on manifolds M and N linear connections are defined being not necessarily metric ones. However, for a special case of metric connections we obtain a basis of differential invariants of the mentioned type. Our considerations are based on new geometric concepts induced by a mapping $f : M \to N$ of manifolds with linear connections, the covariant differentials of the tangent mapping to f, which we introduce for the purposes of this paper on the basis of classical geometrical tools given in [9], [6], [7], [12].

In following two sections we briefly summarize definitions and fundamental properties of dual connection, tensor product connection (sec. 2) and related curvatures (sec. 3) and using general formulas given in [6], [7] we apply them to the case $f : M \to N$. Coordinate expressions for the dual connection and the induced curvature is obtained. The concept of covariant differentials (of general order) of the mapping tangent to f is introduced in section 4 and it is shown that the curvature of corresponding tensor product connection is expressed only by f and by the curvatures of connections defined on manifolds M and N, respectively. In section 5 the relation to geometrical concepts, defined for Riemannian manifolds (see [11]), is discused. Finaly, some applications of general results are presented in section 6, such as a relation to the natural operators and differential invariants of smooth mappings of Riemannian and pseudoriemannian manifolds.

*This research is supported by grants GAČR 201/03/512 of the Czech Grant Agency and MSM 0021622409 of the Ministry of Education, Youth and Sports.

2 Linear connection

Let (E, p, M) be a vector bundle over m-dimensional manifold M. By a *linear connection* on E we mean a linear section $K : E \to J^1 E$. Using the contact morphism $c : J^1 E \to T^* M \otimes_E TE$, such a connection can be viewed as a section

$$K : E \to T^* M \otimes_E TE.$$

Given connection K, K' on E, E' respectively, there is a unique linear connection (*dual connection*) K^* on the vector bundle E^*, and a unique linear connection (*tensor product connection*) $K \otimes K'$ on the vector bundle $E \otimes E'$, such that following diagrams comute

$$\begin{tikzcd}
E \times_M E^* \arrow[r, "\langle\ |\ \rangle"] \arrow[d, "K \times K^*"] & M \times \mathbf{R} \arrow[d, "0 \times \mathrm{id}_{\mathbf{R}}"] \\
J^1 E \times_M J^1 E^* \arrow[r, "J^1\langle\ |\ \rangle"] & T^* M \times \mathbf{R}
\end{tikzcd}
\qquad
\begin{tikzcd}
E \times_M E' \arrow[r, "\otimes"] \arrow[d, "K \times K'"] & E \otimes_M E' \arrow[d, "K \otimes K'"] \\
J^1 E \times_M J^1 E' \arrow[r, "J^1\otimes"] & J^1(E \otimes_M E')
\end{tikzcd}$$

Let (E, p, N) be a vector bundle, $f : M \to N$ be a smooth mapping and $\Delta : E \to T^* E \otimes_E TE$ be a linear connection on E. Let $(f^* E, f^* p, M)$ be the pullback vector bundle of E, $p^* f$ is vector bundle homomorphism given by diagram

$$\begin{tikzcd}
f^* E \arrow[r, "p^* f"] \arrow[d, "f^* p"] & E \arrow[d, "p"] \\
M \arrow[r, "f"] & N
\end{tikzcd} \tag{1}$$

We define $f^* \Delta : f^* E \to T^* M \otimes_{f^* E} T f^* E$ by

$$f^* \Delta(u)(\xi) = (T_u p^* f)^{-1} \circ \Delta(p^* f(u)) \circ (T_{f^* p(u)} f \cdot \xi), \tag{2}$$

for every $\xi \in T_{f^* p(u)} M$ and every $u \in E$. $f^* \Delta$ is a connection on the vector bundle $f^* E$ over M, called the *pullback connection of* Δ.

For our later considerations we give useful coordinate expressions. Let

$$\begin{aligned}
\Gamma :&\quad TM \to T^* M \otimes_{TM} TTM \\
\Delta :&\quad TN \to T^* N \otimes_{TN} TTN \\
K :&\quad E \to T^* M \otimes_E TE \\
K' :&\quad E' \to T^* M \otimes_E TE'
\end{aligned}$$

are linear connection on corresponding vector bundles. Consider local coordinates (x^j) on M, (y^σ) on N. Denote $(x^i, \dot{x}^i)$, $(y^\sigma, \dot{y}^\sigma)$ coordinates on TM, TN, respectively, $1 \le i, j \le m$, $1 \le \sigma, \alpha, \beta \le n$. Denote (x^j, z^α), (x^j, u^β), $(x^j, w^{\alpha\beta})$, coordinates on E, E', $E \otimes E'$ respectively. Using the standard notation, we get the following expressions:

$$\begin{aligned}
K &= \mathrm{d}^l \otimes (\partial_l + K^\alpha_{\gamma l} z^\gamma \partial_\alpha) \\
K' &= \mathrm{d}^l \otimes (\partial_l + K'^\beta_{\delta l} u^\delta \partial_\beta) \\
\Delta &= \mathrm{d}^\sigma \otimes (\partial_\sigma + \Delta^\alpha_{\sigma\beta} \dot{y}^\beta \dot{\partial}_\alpha) \\
\Gamma &= \mathrm{d}^l \otimes (\partial_l + \Gamma^j_{lk} \dot{x}^k \dot{\partial}_j) \\
f^*\Delta &= \mathrm{d}^l \otimes (\partial_l + \Delta^\alpha_{\sigma\beta} f^\sigma_l \dot{y}^\beta \dot{\partial}_\alpha) \\
\Gamma^* &= \mathrm{d}^l \otimes (\partial_l - \Gamma^j_{lk} \dot{x}_j \dot{\partial}^k) \\
K \otimes K' &= \mathrm{d}^i \otimes (\partial_i + (K^\alpha_{\gamma i} w^{\gamma\beta} + K'^\beta_{\delta i} w^{\alpha\delta}) \partial_{\alpha\beta}).
\end{aligned} \tag{3}$$

Let K be a linear connection on E, $s : M \to E$ be a section of E. By a *covariant differential of s with respect to* Γ we mean the section

$$\nabla^K s : M \to T^*M \otimes_M E$$

given by

$$(\nabla^K s)(x)(\xi) = \mathrm{vpr} \circ [(c \circ j^1 s - K \circ s)(x)(\xi)], \tag{4}$$

where $\mathrm{vpr} := \mathrm{pr}_2 \circ \mathrm{vl}^{-1} : VE \to E$ is the vertical projection, $\mathrm{vl} : E \times_M E \to VE$ is canonical vector bundle isomorphism (vertical lift), $c : J^1 E \to T^*M \otimes_E TE$ is the contact morphism.

3 Curvature

Let $K : E \to T^*M \otimes_E TE$ be a linear connection on vector bundle (E, p, M). By a *curvature of* K we mean a section

$$R[K] : E \to \Lambda^2 T^*M \otimes_E VE$$

given by

$$R[K](u)(\xi, \eta) = K(u)[\xi, \eta] - [K(u)(\xi), K(u)(\eta)],$$

for $u \in E$, $\xi, \eta \in \chi(M)$, $[\ ,\ ]$ denotes the Lie bracket. Using the identification $\mathrm{vl} : E \times_M E \to VE$, the curvature can be viewed as a tensor field

$$R[K] : M \to \Lambda^2 T^*M \otimes_M E^* \otimes_M E.$$

In coordinates, using notation from section 2, we get for curvatures corresponding to connections given by (3):

$$\begin{aligned}
R[K] &\in \Omega^2(M, E^* \otimes_M E) \\
R[K'] &\in \Omega^2(M, E'^* \otimes_M E') \\
R[\Delta] &\in \Omega^2(N, (TN)^* \otimes_M TN) \\
R[\Gamma] &\in \Omega^2(M, (TM)^* \otimes_M TM) \\
R[f^*\Delta] &\in \Omega^2(M, (f^*E)^* \otimes_M f^*E) \\
R[\Gamma^*] &\in \Omega^2(M, E \otimes_M E^*) \\
R[K \otimes K'] &\in \Omega^2(M, (E \otimes_M E')^* \otimes_M (E \otimes_M E').
\end{aligned}$$

following local expressions

$$
\begin{aligned}
R[K] &= R[K]^{\alpha}_{\gamma ij} z^{\gamma} \partial_{\alpha} \otimes \mathrm{d}^{i} \wedge \mathrm{d}^{j} \\
R[K'] &= R[K']^{\beta}_{\delta ij} u^{\delta} \partial_{\beta} \otimes \mathrm{d}^{i} \wedge \mathrm{d}^{j} \\
R[\Delta] &= \Omega^{\sigma}_{\nu\eta\omega} \dot{y}^{\nu} \dot{\partial}_{\sigma} \otimes \mathrm{d}^{\eta} \wedge \mathrm{d}^{\omega} \\
R[\Gamma] &= R^{i}_{jkl} \dot{x}^{j} \dot{\partial}_{i} \otimes \mathrm{d}^{k} \wedge \mathrm{d}^{l} \\
R[f^{*}\Delta] &= \Omega^{\sigma}_{\nu\eta\omega} f^{\eta}_{i} f^{\omega}_{j} \dot{y}^{\nu} \dot{\partial}_{\sigma} \otimes \mathrm{d}^{i} \wedge \mathrm{d}^{j} \\
R[\Gamma^{*}] &= -R^{i}_{jkl} \dot{x}_{i} \dot{\partial}^{j} \otimes \mathrm{d}^{k} \wedge \mathrm{d}^{l} \\
R[K \otimes K'] &= (R[K]^{\alpha}_{\gamma ij} w^{\gamma\beta} + R[K']^{\beta}_{\delta ij} w^{\alpha\delta}) \partial_{\alpha\beta} \otimes \mathrm{d}^{i} \wedge \mathrm{d}^{j},
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
R[K]^{\alpha}_{\gamma ij} &= \partial_{j} K^{\alpha}_{\gamma i} - \partial_{i} K^{\alpha}_{\gamma j} + K^{\varepsilon}_{\gamma j} K^{\alpha}_{\varepsilon i} - K^{\varepsilon}_{\gamma i} K^{\alpha}_{\varepsilon j} \\
R[K']^{\beta}_{\delta ij} &= \partial_{j} K'^{\beta}_{\delta i} - \partial_{i} K'^{\beta}_{\delta j} + K'^{\varepsilon}_{\delta j} K'^{\beta}_{\varepsilon i} - K'^{\varepsilon}_{\delta i} K'^{\beta}_{\varepsilon j} \\
R^{i}_{jkl} &= \partial_{l} \Gamma^{i}_{jk} - \partial_{k} \Gamma^{i}_{jl} + \Gamma^{a}_{jl} \Gamma^{i}_{ak} - \Gamma^{a}_{jk} \Gamma^{i}_{al} \\
\Omega^{\sigma}_{\nu\eta\omega} &= \partial_{\omega} \Delta^{\sigma}_{\nu\eta} - \partial_{\eta} \Delta^{\sigma}_{\nu\omega} + \Delta^{\alpha}_{\nu\omega} \Delta^{\sigma}_{\alpha\eta} - \Delta^{\alpha}_{\nu\eta} \Delta^{\sigma}_{\alpha\omega}
\end{aligned}
$$

4 Covariant differentials of a tangent mapping

Let $E^{i} = \otimes^{i}_{M} T^{*}M \otimes_{M} f^{*}TN$ be a vector bundle over M and consider the connection $\otimes^{i}\Gamma^{*} \otimes f^{*}\Delta$ on E^{i}. In coordinates, using (3) and (5) we get local expressions

$$
\begin{aligned}
\otimes^{i}\Gamma^{*} \otimes f^{*}\Delta &= \mathrm{d}^{j} \otimes [\partial_{j} + (\Delta^{\alpha}_{\sigma\beta} f^{\sigma}_{j} w^{\beta}_{k_1\ldots k_i} - \Gamma^{l}_{jk_1} w^{\alpha}_{lk_2\ldots k_i} - \ldots - \Gamma^{l}_{jk_i} w^{\alpha}_{k_1\ldots k_{i-1}l}) \partial^{k_1\ldots k_i}_{\alpha}], \\
R[\otimes^{i}\Gamma^{*} \otimes f^{*}\Delta] &= [\Omega^{\sigma}_{\nu\eta\omega} f^{\eta}_{l} f^{\omega}_{j} w^{\nu}_{k_1\ldots k_i} - R^{a}_{k_1 jl} w^{\sigma}_{ak_2\ldots k_i} - \ldots - R^{a}_{k_i jl} w^{\sigma}_{k_1\ldots k_{i-1}a}] \partial^{k_1\ldots k_i}_{\sigma} \otimes \mathrm{d}^{j} \wedge \mathrm{d}^{l}
\end{aligned}
$$

Let $s : M \to E^{i}$ be a section. In agreement with general definition (4), by the *covariant differential of s with respect to the connections Γ, Δ and a smooth mapping $f : M \to N$*, we mean a section of $\otimes^{i+1}_{M} T^{*}M \otimes_{M} f^{*}TN = E^{i+1}$ given by

$$
\nabla^{(\otimes^{i}\Gamma^{*}\otimes f^{*}\Delta)} s(x)(\xi_1, \ldots, \xi_{i+1}) = \mathrm{vpr} \circ [\mathrm{c} \circ j^{1}s - ((\otimes^{i}\Gamma^{*} \otimes f^{*}\Delta) \circ s)](x)(\xi_1, \ldots, \xi_{i+1}).
$$

Covariant differential may be viewed as a operator $C^{\infty}(E^{i}) \to C^{\infty}(E^{i+1})$.

Consider the tangent mapping $Tf : TM \to TN$. Such mapping determines uniquely a mapping $\mathfrak{T}f : TM \to f^{*}TM$ by $p^{*}f \circ \mathfrak{T}f = Tf$, where $(f^{*}TN, f^{*}p, M)$ is the pullback bundle of TN, and $p^{*}f$ is vector bundle homomorphism given by diagram (1). The mapping $\mathfrak{T}f$ is an element of $C^{\infty}E^{1}$. By a *covariant differential of order r of the tangent mapping Tf* we mean a section $[\nabla^{(\Gamma,\Delta,f)}]^{r}\mathfrak{T}f$ of $C^{\infty}E^{r+1}$, where $[\nabla^{(\Gamma,\Delta,f)}]^{r} : C^{\infty}E^{1} \to C^{\infty}E^{r+1}$ is given by

$$
[\nabla^{(\Gamma,\Delta,f)}]^{r} = \overbrace{\nabla^{(\otimes^{r}\Gamma^{*}\otimes f^{*}\Delta)} \circ \nabla^{(\otimes^{r-1}\Gamma^{*}\otimes f^{*}\Delta)} \ldots \circ \nabla^{(\Gamma^{*}\otimes f^{*}\Delta)}}^{r}.
$$

The chart expression of the covariant differential can be obtained using the coordinate expressions of connections Δ, $f^{*}\Delta$, Γ, Γ^{*} and $\otimes^{i}\Gamma^{*} \otimes f^{*}\Delta$:

Theorem: In coordinates, covariant differential $[\nabla^{(\Gamma,\Delta,f)}]^{k}Tf$ has the following form:

$$
[\nabla^{(\Gamma,\Delta,f)}]^{k}\mathfrak{T}f = K^{\sigma}_{i_1\ldots i_{k+1}} \partial_{\sigma} \otimes \mathrm{d}^{\mathrm{i}_1} \otimes \mathrm{d}^{i_{k+1}},
$$

where coefficients $K^{\sigma}_{i_1\dots i_{k+1}}$ are given by following recurent formulas

$$\begin{aligned}
K^{\sigma}_{i_1} &= f^{\sigma}_{i_1}, \\
K^{\sigma}_{i_1 i_2} &= \partial_{i_2} K^{\sigma}_{i_1} + \Delta^{\sigma}_{\alpha\beta} K^{\alpha}_{i_2} K^{\beta}_{i_1} - \Gamma^{a}_{i_2 i_1} K^{\sigma}_{a}, \\
&\vdots \\
K^{\sigma}_{i_1\dots i_{k+1}} &= \partial_{i_{k+1}} K^{\sigma}_{i_1\dots i_k} + \Delta^{\sigma}_{\alpha\beta} K^{\alpha}_{i_{k+1}} K^{\beta}_{i_1\dots i_k} - \\
&\quad -\Gamma^{a}_{i_{k+1} i_1} K^{\sigma}_{a i_2\dots i_k} - \Gamma^{a}_{i_{k+1} i_2} K^{\sigma}_{a i_1 i_3\dots i_k} - \\
&\quad - \dots - \Gamma^{a}_{i_{k+1} i_k} K^{\sigma}_{a i_1\dots i_{k-1}}.
\end{aligned} \tag{6}$$

5 Geodesics and Riemannian manifolds

Let N be a Riemannian manifold and D^N be a covariant derivative coresponding to the metric connection on N. Consider inclusion $f : M \to N$ of a submanifold $M \subset N$, the induced metric on M, and covariant derivative coresponding to the metric connection D^M. There is an orthogonal decomposition of $T_{f(x)}N$ at any point $f(x) = x \in f(M)$:

$$T_{f(x)}N = T_x M + N_x M,$$

denote $\mathrm{pr}_{TM} : f^*TN \to TM$ and $\mathrm{pr}_{NM} : f^*TN \to NM$ corresponding orthogonal projections, where the set $NM = \cup_{x\in M} N_x M$ considered together with a structure of a vector bundle over M is called *normal bundle*. The following theorem (see [11]) holds.

Theorem: Let $\zeta_N, \eta_N, \nu_N \in C^\infty(TN)$ be any extensions of a sections $\zeta, \eta \in C^\infty(TM)$, $\nu \in C^\infty(NM)$. Than

$$\begin{aligned}
D^N_{\eta_N} \zeta_N &= D^M_\eta \zeta + k_{12}(\eta, \zeta), \\
D^N_{\eta_N} \nu_N &= k_{21}(\eta, \nu) + \nabla_\eta \nu,
\end{aligned}$$

where k_{12} is a symetric bilinear bundle map $TM \times TM \to NM$, k_{21} is a bilinear bundle map $TM \times NM \to TM$, $\nabla_\eta \nu = \mathrm{pr}_{NM} \circ D^N_{\eta_N} \nu_N$ is independent of the extensions and it is a metric derivative on NM.

The objects k_{12}, k_{21} determine each other uniquely and they are both called a *second fundamental form.* Denote g, Δ^g a metric and the metric connection on N, denote Γ^{f^*g} the metric connection on M corresponding to the induced metric f^*g. The relation between the concept of the covariant differential of the mapping tangent to f and the concept of the second fundamental form is described by the following theorem.

Theorem: Let $f : M \to N$ is an inclusion of a Riemannian submanifold. The second fundamental form is the first covariant differential of the mapping tangent to f, i.e.

$$[\nabla^{(\Gamma^{f^*g}, \Delta^g, f)}]\mathfrak{T} f(\xi, \eta) = k_{12}(\xi, \eta).$$

A Riemannian submanifold M of manifold N is called *totally geodesic* if every geodesic in M with initial conditions in (M, TM) is contained in M. An injective immersion $f : M \to N$ of manifolds with connections is called *affine* if for any geodesic α in M, its image $f \circ \alpha$ is a geodesic in N. Every inclusion of totally geodesic Riemannian submanifold is affine.

A necessary and sufficient condition for a Riemannian submanifold to be totally geodesic is vanishing of the second fundamental form. We can generalize this classical result to the case of injective immersion of manifolds with arbitrary connections.

Theorem: An Injective immersion $f : M \to N$ is affine if and only if the first covariant differential of the mapping tangent to f vanishes identicaly.

6 Natural Operators

Riemannian and pseudoriemannian manifolds

Denote Met$M$ the natural bundle of metric fields on M, and $T^*M \otimes TN = \cup_{(x,y)\in M\times N} T^*_x M \otimes T_y N$ with structure of a fibered bundle over $M \times N$. We consider a first order natural bundle F:

$$F(M,N) = T^*M \otimes TN \times_M \mathrm{Met}M \times_N \mathrm{Met}N$$

over $M \times N$ isomorphic to $(P^1M \times P^1N)[Q, G^1_m \times G^1_n]$ with typical fiber

$$Q = \mathbf{R}^{m*} \otimes \mathbf{R}^n \times \mathrm{reg}(\mathbf{R}^{m*} \odot \mathbf{R}^{m*}) \times \mathrm{reg}(\mathbf{R}^{n*} \odot \mathbf{R}^{n*}).$$

Using the reduction $W^k(P^1M \times P^1N) \to P^{k+1}M \times P^{k+1}N$ to the subgroup $G^{k+1}_m \times G^{k+1}_n \subset W^k(G^1_m \times G^1_n)$, its prolongation is a fiber bundle $F^k(M,N) = (P^{k+1}M \times P^{k+1}N)[T^k_{m+n}Q, G^{k+1}_m \times G^{k+1}_n]$.

Consider a sheaf of sections $\Phi_{M,N} \subset C^\infty(F(M,N), M\times N)$ defined by $s \in \Phi \Leftrightarrow s = (f, g \times h)$, where $h \in C^\infty(\mathrm{Met}M, M)$, $g \in C^\infty(\mathrm{Met}N, N)$, and $f \in C^\infty(T^*M \otimes TN, M \times N)$ is a section constant on N. Define *k-th order natural operator of smooth mappings of manifolds with metric fields* $D : F \to G$ as a system of smooth local operators $D_{M,N} : \Phi \to C^\infty(G(M,N), M \times N)$ satisfying:

1. $D_{(M',N')}(F(f,g) \circ s \circ (f^{-1}, g^{-1})) = G(f,g) \circ D_{(M,N)} s \circ (f^{-1}, g^{-1})$, $\forall s \in \Phi$, $\forall (f,g) \in \mathrm{Mor}\mathcal{M}f_m \times \mathcal{M}f_n$,
2. $j^k s_1 = j^k s_2 \Rightarrow D_{M,N}(s_1) = D_{M,N}(s_2)$, $\forall s_1, s_2 \in \Phi$.

Consider a manifold

$$Q^k = T^k_m(\mathbf{R}^{m*} \otimes \mathbf{R}^n) \times T^k_m \mathrm{reg}(\mathbf{R}^{m*} \odot \mathbf{R}^{m*}) \times T^k_n \mathrm{reg}(\mathbf{R}^{n*} \odot \mathbf{R}^{n*}),$$

which is closed under the action of $G^{k+1}_m \times G^{k+1}_n$. Bundle

$$F^k_0(M,N) = (P^{k+1}M \times P^{k+1}N)[Q^k, G^{k+1}_m \times G^{k+1}_n]$$

is a subbundle of $F^k(M,N)$. Natural operators of smooth mappings of manifolds with metric fields with values in a natural bundle of order p, are in bijective correspondence with differential invariants of corresponding type fibers $Q^k \to S$ (where S is a left $G^p_m \times G^p_n$ manifold, the typical fiber of natural bundle G), (see [15]). Using the orbit reduction method, we obtain the basis of invariants for the case $p = 1$ (see [13]).

Theorem: The basis of k-th order diferential invariants of smooth mappings f of manifolds with metric fields is formed by the components of the metric fields on both manifolds, by the components of curvature tensors corresponding to the metric connection on both manifolds and its covariant derivatives up to the order $k-2$, the components of tangent mapping to f and the components $K^\sigma_{i_1 \ldots i_l}$, $2 \le l \le k+1$ of the covariant differential of the tangent mapping to f up to the order k.

Manifolds with connections

Denote Cla$M$ the natural bundle of classical connection on M. We consider a second order natural bundle F:

$$F(M,N) = J^2(M,N) \times_M \mathrm{Cla}M \times_N \mathrm{Cla}N$$

over $M \times N$ isomorphic to $(P^2M \times P^2N)[S, G_m^2 \times G_n^2]$ with typical fiber

$$Q = S^2\mathbf{R}^{m*} \otimes \mathbf{R}^n \times (\mathbf{R}^m \otimes S^2\mathbf{R}^{m*}) \times (\mathbf{R}^n \otimes S^2\mathbf{R}^{n*}).$$

Using the reduction $W^k(P^2M \times P^2N) \to P^{k+2}M \times P^{k+2}N$ to the subgroup $G_m^{k+2} \times G_n^{k+2} \subset W^k(G_m^2 \times G_n^2)$, its prolongation is a fiber bundle $F^k(M,N) = (P^{k+2}M \times P^{k+2}N)[T_{m+n}^k Q, G_m^{k+2} \times G_n^{k+2}]$. Consider a sheaf of sections $\Phi_{M,N} \subset C^\infty(F(M,N), M \times N)$ defined by $s \in \Phi \Leftrightarrow s = (f, \Gamma \times \Delta)$, where $\Gamma \in C^\infty(\mathrm{Cla}M, M)$, $\Delta \in C^\infty(\mathrm{Cla}N, N)$, and $f \in C^\infty(J^2(M,N), M \times N)$ is a section constant on N. Define *k-th order natural operator of smooth mappings of manifolds with connections* $D : F \to G$ as a system of smooth local operators $D_{M,N} : \Phi \to C^\infty(G(M,N), M \times N)$ satisfying:

1. $D_{(M',N')}(F(f,g) \circ s \circ (f^{-1}, g^{-1})) = G(f,g) \circ D_{(M,N)} s \circ (f^{-1}, g^{-1})$, $\forall s \in \Phi$, $\forall (f,g) \in$ Mor$\mathcal{M}f_m \times \mathcal{M}f_n$,
2. $j^k s_1 = j^k s_2 \Rightarrow D_{M,N}(s_1) = D_{M,N}(s_2)$, $\forall s_1, s_2 \in \Phi$.

Consider a manifold

$$Q^k = T_m^k(S^2\mathbf{R}^{m*} \otimes \mathbf{R}^n) \times T_m^k(\mathbf{R}^m \otimes S^2\mathbf{R}^{m*}) \times T_n^k(\mathbf{R}^n \otimes S^2\mathbf{R}^{n*}),$$

which is closed under the action of $G_m^{k+2} \times G_n^{k+2}$. Bundle

$$F_0^k(M,N) = (P^{k+1}M \times P^{k+1}N)[Q^k, G_m^{k+1} \times G_n^{k+1}]$$

is a subbundle of $F^k(M,N)$.

Natural operators of smooth mappings of manifolds with connections with values in a natural bundle of order p, are in bijective correspondence with differential invariants of corresponding type fibers $Q^k \to S$ (where S is a left $G_m^p \times G_n^p$ manifold, the typical fiber of natural bundle G), (see [15]). Using the orbit reduction method we obtain the basis of invariants for the case $p = 1$.

Theorem: The basis of k-th order diferential invariants of smooth mappings f of manifolds with classical connections is formed by the components of the corresponding curvature tensors on both manifolds and its covariant derivatives up to the order $k-1$, the componetns of the tangent mapping to f and the components $K^\sigma_{i_1 \ldots i_l}$, $2 \le l \le k+2$ of the covariant differential of the tangent mapping to f.

References

[1] M. Atiyah, R. Bott, V. K. Patodi: *On the Heat Equation and the Index Theorem*, Inventiones Math. **19** (1973) 279—330, Errata **28** (1975) 277—280.

[2] J. Dieudonné: *Éléments d'Analyse*, Tome III, Gauthier-Villars, Paris 1974.

[3] P. B. Gilkey: *Curvature and the Eigenvalues of the Laplacian for Elliptic Complexes*, Advances in Math. **10** (1973) 344-382.

[4] G. B. Gurevich: *Foundation of the Theory of Algebraic Invariants*, Groningen 1964.

[5] P. Hořava, *Topological Rigid String Theory and Two Dimensional QCD*, Nucl. Phys. **B463** (1996) 238-286.

[6] J. Janyška, *Reduction Theorems for General Linear Connection*, Differential Geometry and Its Applications **20** (2004) 177-196.

[7] J. Janyška, *On the Curvature of Tensor Product Connections and Covariant Differentials*, Supplemento di Rendiconti del Circolo Matematico di Palermo, Serie II - No 72 - (2004) 135-143.

[8] J. Jost: *Riemannian Geometry and Geometric Analysis*, Second Edition, Springer-Verlag, Berlin 1998.

[9] I. Kolář, P. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin 1993.

[10] D. Krupka, J. Janyška: *Lectures on Differential Invariants*, J. E. Purkyně University, Brno 1990.

[11] S. Lang:*Fundamentals of Differential Geometry*, Springer Verlag, Berlin 2001.

[12] L. Mangiarotti, M. Modugno:*Connections and Differential Calculus on Fibred Manifolds. Applications to Field Theory.*, preprint Istituto di Mathematica Applicata "G. Sansone", Florence 1989.

[13] P. Musilová, J. Musilová: *Differential Invariants of Immersions of Manifolds with Metric Fields*, Communications in Mathematical Physics (2004), DOI: 10.1007/s00220-004-1113-8.

[14] P. Musilová, D. Krupka: *Differential Invariants of Immersions of Manifolds with Metric Fields*, Reports on Mathematical Physics**51** (2003)

[15] P. Musilová, J. Musilová:*Natural operators of smooth mapings of manifolds with metric fields*, Reports on Mathematicl Physics, in print.

[16] H. Weyl: *The Classical Groups, Their Invariants and Representations*, Princeton University, Princeton 1997.