NATURAL OPERATORS OF SMOOTH MAPPINGS OF MANIFOLDS WITH METRIC FIELDS*

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In this paper we introduce the concepts of both a natural bundle and a natural operator generalized for the case of the category $\mathcal{M}f_m \times \mathcal{M}f_n$ of cartesian products of two manifolds and products of local diffeomorphisms. It is shown, that any r-th order natural bundle over $M \times N$ has a structure of an associated bundle $(P^r M \times P^r N)[S, G^r_m \times G^r_n]$. We consider prolongations of such associated bundles and their reduction with respect to a chosen subgroup. The existence of a bijective correspondence between natural operators of the order k and the equivariant mappings of corresponding type fibers is proved. A basis of invariants of arbitrary order is constructed for natural operators of smooth mappings of manifolds endowed with metric fields or connections, with values in a natural bundle of order one.

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1. Natural bundles and operators

In the whole paper we follow by analogy the standard notation as introduced in [9]. Consider the category $\mathcal{M}f_m \times \mathcal{M}f_n$, the objects of which are pairs (M, N) of smooth manifolds of dimensions m, n and its morphisms are pairs of local diffeomorphisms $f : M \to M'$, $g : N \to N'$. A *natural bundle* over $\mathcal{M}f_m \times \mathcal{M}f_n$ is defined as a covariant functor $F : \mathcal{M}f_m \times \mathcal{M}f_n \to \mathcal{FM}$, where $\mathcal{FM}$ is a category of fibered manifolds and fibered morphisms, satisfying the following conditions:

(i) $B \circ F = \mathrm{Id}_{\mathcal{M}f_m \times \mathcal{M}f_n}$ where $B : \mathcal{FM} \to \mathcal{M}f$ is the base functor, i.e. to any pair (M, N) of smooth manifolds a fibered manifold over $M \times N$ is assigned.

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(ii) If $i : U \to M$, $j : V \to N$ are inclusions of open submanifolds, then $F(U,V) = p^{-1}_{(M,N)}(U,V)$ and $F(i,j)$ is the inclusion of $p^{-1}_{(M,N)}(U,V)$ into $F(M,N)$, $p_{(M,N)}$ denote the projection of a fibered manifold $F(M,N)$.

Remark: As a consequence of this definition we can formulate the regularity condition: smoothly parametrized systems of pairs of local diffeomorphisms are transformed into smoothly parametrized systems of fibered local isomorphisms.

A natural bundle $F : \mathcal{M}f_m \times \mathcal{M}f_n \to \mathcal{FM}$ is said to be *of the order* r, $0 \le r < \infty$, if for all pairs of local diffeomorphisms (f,g), and (f',g') $(f, f' : M \to M'$, $g, g' : N \to N')$ and any point $(x,y) \in M \times N$ the condition $j^r_{(x,y)}(f \times g) = j^r_{(x,y)}(f' \times g')$ implies $F(f,g)|_{F_{(x,y)}(M,N)} = F(f',g')|_{F_{(x,y)}(M,N)}$.

A *natural operator* $D : F \to G$ between two natural bundles F and G is defined as a system of smooth local operators $D_{(M,N)} : C^\infty(F(M,N)) \to C^\infty(G(M,N))$, $(M,N) \in \mathrm{Ob}(\mathcal{M}f_m \times \mathcal{M}f_n)$, satisfying

$$D_{(M',N')}(F(f,g) \circ s \circ (f^{-1}, g^{-1})) = G(f,g) \circ D_{(M,N)} s \circ (f^{-1}, g^{-1}) \qquad (1)$$

for every section $s \in C^\infty(F(M,N))$ and every pair of diffeomorphisms $f : M \to M'$, $g : N \to N'$.

A natural operator $D : F \to G$ is said to be *of the order* k, if all operators $D_{(M,N)}$ are of the order k (i.e. the condition $j^k_{(x,y)} s_1 = j^k_{(x,y)} s_2$ implies $D_{(M,N)}(s_1)(x,y) = D_{(M,N)}(s_2)(x,y)$).

Theorem 1. *Any r-th order natural bundle over (M,N) has a structure of an associated fibered bundle $(P^rM \times P^rN)[S, G^r_m \times G^r_n]$.*

Proof: Consider a natural bundle $F : \mathcal{M}f_m \times \mathcal{M}f_n \ni (M,N) \to F(M,N) \in \mathcal{FM}$ with a standard fibre S and smooth mappings $F_{(M,N),(M',N')}$ associated to the bundle functor: $F_{(M,N),(M',N')}$:

$$\mathrm{inv}J^r(M,M') \times \mathrm{inv}J^r(N,N') \times_{(M,N)} F(M,N) \to F(M',N'),$$

$$F_{(M,N),(M',N')}(j^r_x f, j^r_y g, Z) = F(f,g)(Z).$$

Recall that $G^r_m = \mathrm{inv}J^r_0(\mathrm{R}^m, \mathrm{R}^m)_0 \subset \mathrm{inv}J^r(\mathrm{R}^m, \mathrm{R}^m)$ and $P^rM = \mathrm{inv}J^r_0(\mathrm{R}^m, M) \subset \mathrm{inv}J^r(\mathrm{R}^m, M)$ where $\mathrm{inv}J^r_0(\mathrm{R}^m, M)$ denotes invertible r-jets with the source at the point $0 \in \mathrm{R}^m$.

Now we have the induced action

$$l = F_{(\mathrm{R}^m,\mathrm{R}^n),(\mathrm{R}^m,\mathrm{R}^n)}|_{G^r_m \times G^r_n \times S} : G^r_m \times G^r_n \times S \to S,$$

and the induced mapping

$$q_{(M,N)} = F_{(\mathrm{R}^m,\mathrm{R}^n),(M,N)}|_{P^rM\times P^rN\times S} : P^rM \times P^rN \times S \to F(M,N).$$

We will show that the mapping $q_{(M,N)}$ defines a structure of the associated fiber bundle $(P^rM \times P^rN)[S, G^r_m \times G^r_n]$ on $F(M,N)$: It is evident that for any $\zeta = (j^r_0 f, j^r_0 g) \in (P^rM \times P^rN)$, $s \in S$ and $a = (J^r_0\alpha, j^r_0\gamma) \in G^r_m \times G^r_n$ it holds

$$q_{(M,N)}(\zeta \circ a, l(a^{-1}, s)) = q_{(M,N)}(\zeta, s),$$

and that mappings $(q_{(M,N)})_\zeta := q_{(M,N)}(\zeta, \cdot) : S \to S$ are diffeomorphisms. For any pair of local diffeomorphisms $f : M \to M'$, $g : N \to N'$ and for an arbitrary $\zeta = (j^r_0\mu, j^r_0\nu) \in P^rM \times P^rN$ we have

$$F(f,g) \circ q_{(M,N)}(\zeta, s) = q_{(M',N')}((j^r_0(f \circ \mu), j^r_0(g \circ \nu)), s).$$

Thus $F(f,g)$ corresponds to $[P^rf \times P^rg, \mathrm{id}_S] : (P^rM \times P^rN)[S, G^r_m \times G^r_n] \to (P^rM \times P^rN)[S, G^r_m \times G^r_n]$.

♡

THEOREM 2. *Let F and G be natural bundles. There is a bijective correspondence between k-th order natural operators $D : F \to G$ and natural transformations $T : J^k \circ F \to G$.*

PROOF: Consider a natural transformation $T : J^k \circ F \to G$. Define $D_{(M,N)} : C^\infty(F(M,N)) \to C^\infty(G(M,N))$ by $D_{(M,N)}(s(x,y)) = T_{(M,N)}(j^k_{(x,y)}s)$ for all $s \in C^\infty(F(M,N))$. The maps $D_{(M,N)}$ are smooth fibered morphisms over the identity $\mathrm{id}_{M\times N}$. So we need to prove the locality and (1). By the definition of natural transformation we have for all morphisms $(f,g) : (M,N) \to (M',N')$:

$$T_{(M',N')} \circ ((J^k \circ F)(f,g)) = G(f,g) \circ T_{(M,N)}.$$

The locality follows immediately. For diffeomorphisms $f : M \to M'$, $g : N \to N'$ we obtain

$$D_{(M',N')}(F(f,g) \circ s \circ (f^{-1}, g^{-1})(f(x), g(y)) =$$

$$= T_{(M',N')}(j^k_{(f(x),(g(y))}(F(f,g) \circ s \circ (f^{-1}, g^{-1})) = T_{(M',N')}[(j^kF)(f,g)(j^k_{(x,y)}s)] =$$

$$= G(f,g) \circ T_{(M,N)}(j^k_{(x,y)}s) = G(f,g) \circ D_{(M,N)}(s(x,y)) =$$

$$= G(f,g) \circ D_{(M,N)}(s) \circ (f^{-1}, g^{-1})(f(x), g(y)),$$

which proves that (1) is satisfied.

On the other hand, let $\mathcal{D}_{(M,N)} : J^k \circ F(M,N) \to G(M,N)$ be associated maps to a k-th order natural operator $D : F \to G$ defined by $\mathcal{D}_{(M,N)}(j^k_{(x,y)}s) = D_{(M,N)}(s(x,y))$. We show that they generate a natural transformation $T : J^k \circ F \to G$, i.e. $G(f,g) \circ \mathcal{D}_{(M,N)} = \mathcal{D}_{(M',N')} \circ J^k \circ F(f,g)$ for any pair of local diffeomorphisms $f : M \to M'$, $g : N \to N'$.

$$\mathcal{D}_{M',N'}((J^k F(f,g)(j^k_{(x,y)}s)) =$$

$$= \mathcal{D}_{(M',N')}(j^k(F(f,g) \circ s \circ (f^{-1}, g^{-1}))(f(x), g(y))) =$$

$$= D_{(M',N')}(F(f,g) \circ s \circ (f^{-1}, g^{-1})(f(x), g(y)) =$$

$$= G(f,g) \circ D_{(M,N)} s(x,y) = G(f,g) \circ \mathcal{D}_{(M,N)}(j^k_{(x,y)}s).$$

♡

THEOREM 3. *There is a bijective correspondence between the set of all natural transformations between two r-th order natural bundles over $\mathcal{M}f_m \times \mathcal{M}f_n$ and the set of smooth $G^r_m \times G^r_n$-equivariant maps between their standard fibers.*

PROOF: Let P, Q be standard fibers of natural bundles F, G respectively, and let $\tau : P \to Q$ be a smooth $G^r_m \times G^r_n$-equivariant mapping. We define natural transformation by

$$T_{(M,N)} = \{\mathrm{id}_{P^r M \times P^r N}, \tau\} : (P^r M \times P^r N)[P, G^r_m \times G^r_n] \to$$
$$\to (P^r M \times P^r N)[Q, G^r_m \times G^r_n].$$

Conversely, consider natural bundles F, G with standard fibers P, Q respectively, and a natural transformation $T : F \to G$. Taking into account the locality of natural bundles and the fact that any such natural transformation covers the identity we can conclude that T is determined by its restriction to the standard fiber, which is equivariant with respect to the induced action.

♡

THEOREM 4. *Let F be a natural bundle of order r with a typical fiber S. Then k-th jet prolongation of F is a natural bundle of order $r+k$, and the typical fiber of $J^k F$ is $T^k_{m+n}S = J^k_0(\mathrm{R}^{m+n}, S)$, where m resp. n are dimensions of underlying manifolds.*

PROOF: It is evident that axioms for a natural bundle are satisfied. $J^k(F(f,g))$ depends on $(k+r)$-jets of f, g in the underlying points only. We can identify $C^\infty(F(\mathrm{R}^m \times \mathrm{R}^n))$ with $C^\infty(\mathrm{R}^{n+m}, S)$ so that the typical fiber of $J^k F$ is $T^k_{m+n}S$.

♡

As a consequence of Theorems 2, 3 and 4 we have the following result:

THEOREM 5. *Let F, F' be natural bundles of order r, r' respectively, with typical fibers S resp. S'. There is a bijective correspondence between the set of all k-th order natural operators $D : F \to F'$ and the set of all smooth $G^q_m \times G^q_n$-equivariant mappings (differential invariants) $A : T^k_{m+n}S \to S'$, where $q = \max(r+k, r')$.*

REMARK: All general theorems and lemmas, which are well-known from the theory of natural operators (see [9]) can be reformulated for the case of natural bundles and operators over pairs of underlying manifolds.

Consider a Lie group G. Its (d,k)-principal prolongation is a semi-direct product $W^k_d G = G^k_d \times T^k_d G$. Let P be a principal bundle with the structure group G and the base manifold D, $\dim D = d$. Its prolongation $W^k P = P^k D \times_D J^k P$ is a principal bundle over the same base manifold with the structure group $W^k_d G$. According to the general theory (see [9], [10]), the following proposition holds:

PROPOSITION 6. *The k-th jet prolongation $J^k F$ of any associated fiber bundle $F = P[S,G]$ (with principal bundle P, structure group G, typical fibre S and the dimension of base manifold $\dim D{=}d$) has a canonical structure of the associated bundle $W^k P[T^k_d S, W^k_d G]$.*

PROPOSITION 7. *There is a natural inclusion of $G^{r+k}_m \times G^{r+k}_n$ into the group $W^k_{m+n}(G^r_m \times G^r_n)$.*

PROOF: Consider a natural inclusion $T_m : G^{r+k}_m \to W^k_m(G^r_m) = G^k_m \times T^k_m G^r_m$ (see [9], [10]). There is an inclusion

$$T : W^k_m(G^r_m) \times W^k_n(G^r_n) \to W^k_{m+n}(G^r_m \times G^r_n)$$

given by

$$(j^k_{0_{\mathbb{R}^m}}\alpha, j^k_{0_{\mathbb{R}^m}}\xi), (j^k_{0_{\mathbb{R}^n}}\gamma, j^k_{0_{\mathbb{R}^n}}\zeta) \to (j^k_{0_{\mathbb{R}^{m+n}}}(\alpha \times \gamma), j^k_{0_{\mathbb{R}^{m+n}}}(\xi \times \zeta)).$$

Thus, $T \circ (T_m \times T_n)$ is an inclusion of the group $G^{r+k}_m \times G^{r+k}_n$ into the group $W^k_{m+n}(G^r_m \times G^r_n)$.

♡

THEOREM 8. *The principal fiber bundle $P^{k,r}_0 = P^{r+k}M \times P^{r+k}N$ is a reduction of the principal fiber bundle $W^k(P^r M \times P^r N)$ with respect to the subgroup $G^{r,k}_0 = G^{r+k}_m \times G^{r+k}_n \subset W^k_{m+n}(G^r_m \times G^r_n)$.*

PROOF: By classical results, there is a reduction

$$\iota_m : P^{r+k}M \to W^k(P^rM) = P^kM \times_M J^kP^rM$$

to the structure group G_m^{r+k}. Consider an inclusion

$$\iota : W^k(P^rM) \times W^k(P^rN) \to W^k(P^rM \times P^rN)$$

given by

$$\begin{aligned}&\iota[(j^k_{0_{\mathrm{R}^m}}\zeta_0, j^k\zeta_1(\zeta_0(0_{\mathrm{R}^m}))), (j^k_{0_{\mathrm{R}^n}}\xi_0, j^k\xi_1(\xi_0(0_{\mathrm{R}_n})))] =\\ &= (j^k_{0_{\mathrm{R}^{m+n}}}(\zeta_0 \times \xi_0), j^k(\zeta_1 \times \xi_1)(\zeta_0(0_{\mathrm{R}^m}), \xi_0(0_{\mathrm{R}^n}))).\end{aligned}$$

Clearly, ι is a homomorphism of principal bundles over the identity, so it is a reduction of $W^k(P^rM \times P^rN)$ to the structure group $W_m^k(G_m^r) \times W_n^k(G_n^r)$. Consequently $(\iota_m \times \iota_n) \circ \iota$ is reduction in consideration.

♡

2. Natural operators of smooth mapping of manifolds with connection or metric fields

We use the orbit reduction method to compute the basis of natural operators of smooth mapping of manifolds with metric fields. Analogical problem is formulated and solved in [17]. Here we reformulate it in terms of natural bundles and operators and summarize the main results.

Let M, N are smooth manifolds of dimensions m, n respectively. Consider a natural bundle

$$F(M,N) = T^*M \otimes TN \times_M \mathrm{Met}M \times_N \mathrm{Met}N$$

over $M \times N$ with the typical fiber

$$S = \mathrm{R}^{m*} \otimes \mathrm{R}^n \times \mathrm{reg}(\mathrm{R}^{m*} \odot \mathrm{R}^{m*}) \times \mathrm{reg}(\mathrm{R}^{n*} \odot \mathrm{R}^{n*}).$$

The k-th prolongation is isomorphic with the asociated fibre bundle

$$W^k(P^1M \times P^1N)[T^k_{m+n}S, G^1_m \times G^1_n].$$

We are interested in natural operators defined on a sheaf of smooth sections of natural bundle $F'(M,N) = (P^{k+1}M \times P^{k+1}N)[S^k, G_m^{k+1} \times G_n^{k+1}]$, with the typical fiber

$$S^k = T^k_m(\mathrm{R}^{m*} \otimes \mathrm{R}^n) \times T^k_m\mathrm{reg}(\mathrm{R}^{m*} \odot \mathrm{R}^{m*}) \times T^k_n\mathrm{reg}(\mathrm{R}^{n*} \odot \mathrm{R}^{n*}).$$

Clearly $S^k \subset T^k_{m+n}S$ is a submanifold, which is closed under the action of $G_m^{k+1} \times G_n^{k+1}$. Thus, such a restriction, which is moreover physically meaningful, is correct. The bundle $F'(M,N)$ is a subbundle of $J^kF(M,N)$. By theorems in section 1, natural operators

$D : F' \to G$, where G is a natural bundle of order one, are in bijective correspondence with equivariant maps (differential invariants)

$$I : S^k \to S_0,$$

where S_0 is a typical fiber of natural bundle G, i.e. a left $G^1_m \times G^1_n$ manifold.

Denote

$$f^\sigma_i, h_{ij}, g_{\sigma\nu}, \quad 1 \leq i \leq j \leq m, 1 \leq \sigma \leq \nu \leq n$$

the canonical coordinates on S, and

$$f^\sigma_{i_1 \dots i_{p+1}}, h_{ij,l_1 \dots l_p}, g_{\sigma\nu,\eta_1 \dots \eta_p}, \quad p = 0, \dots k$$

the canonical coordinates on S^k.

We introduce the coordinates *adapted* to the action of $G^{k+1}_m \times G^{k+1}_n$:

$$\begin{gathered} h_{ij}, \Gamma^i_{jp,l_1 \dots l_t}, R_{ijpl;m_1;\dots;m_s}, \\ g_{\sigma\nu}, \Delta^\sigma_{\nu\eta\gamma_1 \dots \gamma_t}, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_s}, \\ K^\sigma_{i_1 \dots i_q}, \\ 0 \leq s \leq r-2, 0 \leq t \leq k-1, 1 \leq q \leq k+1 \end{gathered}$$

where

$$(h_{ij}, \Gamma^i_{jp,l_1 \dots l_t}, R_{ijpl;m_1;\dots;m_s})$$

are well-known coordinates on $T^k_m \mathrm{MetR}^m$ adapted to the action of G^{k+1}_m. These coordinates represent the metric tensor, the connection, the curvature tensor and its covariant derivatives.

Analogously

$$(g_{\sigma\nu}, \Delta^\sigma_{\nu\eta,\gamma_1 \dots \gamma_t}, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_s})$$

are coordinates on $T^k_n \mathrm{MetR}^n$ adapted to the action of G^{k+1}_n.

Moreover, $K^\sigma_{i_1 \dots i_p}$ are given by transformation equations

$$\begin{aligned} K^\sigma_{i_1} &= f^\sigma_{i_1}, \\ K^\sigma_{i_1 i_2} &= K^\sigma_{i_1,i_2} + \Delta^\sigma_{\alpha\beta} K^\alpha_{i_2} K^\beta_{i_1} - \Gamma^a_{i_2 i_1} K^\sigma_a, \\ &\vdots \\ K^\sigma_{i_1 \dots i_{k+1}} &= K^\sigma_{i_1 \dots i_k, i_{k+1}} + \Delta^\sigma_{\alpha\beta} K^\alpha_{i_{k+1}} K^\beta_{i_1 \dots i_k} - \\ &\quad - \Gamma^a_{i_{k+1} i_1} K^\sigma_{a i_2 \dots i_k} - \Gamma^a_{i_{k+1} i_2} K^\sigma_{a i_1 i_3 \dots i_k} - \\ &\quad - \dots - \Gamma^a_{i_{k+1} i_k} K^\sigma_{a i_1 \dots i_{k-1}}. \end{aligned} \tag{2}$$

Let I_s be a multiindex of length z, $z \in \{2, \dots, k+1\}$. We can write (2) in a form

$$K^{\sigma}_{I_{z-1}i_z} = [K^{\sigma}_{I_{z-1},i_z} + \Delta^{\sigma}_{\alpha\beta} K^{\alpha}_{i_z} K^{\beta}_{I_{z-1}} - \sum_{i \in I_{z-1}} \Gamma^{a}_{i_z i} K^{\sigma}_{a(I_{z-1}\setminus\{i\})}].$$

We should specify the notation K_{I_{z-1},i_z}. Let $K = K(h_{ij,I_t}, g_{\sigma\nu,\Upsilon_t}, f^{\sigma}_{I_{t+1}})$, where I_t, Υ_t are multiindices of length t, $t \in \{0, \dots, r-1\}$. By $K_{,u}$ we denote the function given by

$$K_{,u} = \sum_{t=0}^{r-1} \left(\frac{\partial K}{\partial h_{ij,I_t}} h_{ij,I_t u} + \frac{\partial K}{\partial g_{\sigma\nu,\Upsilon_t}} g_{\sigma\nu,\Upsilon_t \eta} f^{\eta}_{u} + \frac{\partial K}{\partial f^{\sigma}_{I_{t+1}}} f^{\sigma}_{I_{t+1}u} \right).$$

For the action of $G^{k+1}_m \times G^{k+1}_n$ resp. $K^{k+1,1}_m \times K^{k+1,1}_n$ and $p \in \{1, \dots, k+1\}$ we get

$$\overline{K}^{\sigma}_{i_1 \dots i_p} = a^{j_1}_{i_1} \dots a^{j_p}_{i_p} d^{\sigma}_{\nu} K^{\nu}_{j_1 \dots j_p}$$

resp.

$$\overline{K}^{\sigma}_{i_1 \dots i_p} = K^{\sigma}_{i_1 \dots i_p}. \tag{3}$$

Using the orbit reduction method (see [9]) and the fact that the basis of invariants from metric field is formed by metric tensor components and curvature tensor components and its covariant derivatives only (see [1]) we have the following theorem (see [17]).

THEOREM 9. *The set $S^k/(K^{k+1,1}_m \times K^{k+1,1}_n)$ has the orbit manifold structure, given by the global coordinate chart with coordinates* $(h_{ij}, R_{ijpl}, R_{ijpl;m_1}, \dots, R_{ijpl;m_1;\dots;m_{k-2}}, g_{\sigma\nu}, \Omega_{\sigma\nu\eta\omega}, \Omega_{\sigma\nu\eta\omega;\gamma_1}, \dots, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_{k-2}}, K^{\sigma}_{i_1}, \dots, K^{\sigma}_{i_1 \dots i_{k+1}})$.

The coordinates $(h_{ij}, R_{ijpl}, R_{ijpl;m_1}, \dots, R_{ijpl;m_1;\dots;m_{k-2}}, g_{\sigma\nu}, \Omega_{\sigma\nu\eta\omega}, \Omega_{\sigma\nu\eta\omega;\gamma_1}, \dots, \Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_{k-2}}, K^{\sigma}_{i_1}, \dots, K^{\sigma}_{i_1 \dots i_{k+1}})$ form a basis of k-th order differential invariants.

3. Geometrical interpretation

For our generalization of concepts of a natural bundle and a natural operator to the case of category $\mathcal{M}f_m \times \mathcal{M}f_n$ we now present a geometrical interpretation (in detail see [17]). It is based on connections on vector bundles presented in [6], [7] and [16] and we recall them briefly.

Consider a linear connection Γ on a vector bundle (E, p, M), i.e. a linear section $\Gamma : E \to J^1 E$. We denote by Γ^* the dual connection $\Gamma^* : E^* \to J^1 E^*$. For vector bundles (E, p, M) and (E', p', M) over M we denote by $\Gamma \otimes \Gamma' : E \otimes_M E' \to J^1(E \otimes_M E')$ the tensor product connection of Γ and Γ'. These two type connections are uniquely defined by the commutativity of diagrams

$$\begin{tikzcd} E \times_M E^* \arrow[r, "\langle \ | \ \rangle"] \arrow[d, "\Gamma \times \Gamma^*"] & M \times \mathbf{R} \arrow[d, "0 \times \mathrm{id}_{\mathbf{R}}"] \\ J^1E \times_M J^1E^* \arrow[r, "J^1\langle \ | \ \rangle"] & T^*M \times \mathbf{R} \end{tikzcd} \qquad \begin{tikzcd} E \times_M E' \arrow[r, "\otimes"] \arrow[d, "\Gamma \times \Gamma'"] & E \otimes_M E' \arrow[d, "\Gamma \otimes \Gamma'"] \\ J^1E \times_M J^1E' \arrow[r, "J^1\otimes"] & J^1(E \otimes_M E') \end{tikzcd}$$

Considering the contact mapping $c:\ J^1E \hookrightarrow T^*M \otimes_M TE$, the linear connection Γ can be regarded as a TE-valued 1-form $\Gamma:\ E \to T^*M \otimes TE$ projecting on id_{TE}.

Let (E, p, N) be a vector bundle and let $f:\ M \to N$ be a smooth mapping. Consider a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r, "p^*f"] \arrow[d, "f^*p"] & E \arrow[d, "p"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

in which (f^*E, f^*p, M) is the pullback bundle and p^*f is the vector bundle homomorphism. Moreover, p^*f is a fiberwise diffeomorphism. It is easy to prove that the mapping $f^*\Delta:\ f^*E \ni u \to f^*\Delta(u) \in T^*M \otimes_M Tf^*E$ given by

$$f^*\Delta(u)(\xi) = (T_u p^* f)^{-1} \circ \Delta(p^*f(u)) \circ (T_{f^*p(u)} f \cdot \xi), \tag{4}$$

for every $\xi \in T_{f^*p(u)}M$ is a connection on the vector bundle f^*E over M, called the *pull-back connection of* Δ. Consider local coordinates (x^j) on M, (y^σ) on N, and $(x^j, w^\alpha_{k_1 \dots k_i})$ on $\otimes^i T^*M \otimes f^*TN$. Using the standard notation, we get the following expressions:

$$\begin{aligned} \Delta &= \mathrm{d}^\sigma \otimes (\partial_\sigma + \Delta^\alpha_{\sigma\beta} \dot{y}^\beta \dot{\partial}_\alpha) \\ \Gamma &= \mathrm{d}^l \otimes (\partial_l - \Gamma^j_{lk} \dot{x}^k \dot{\partial}_j) \\ f^*\Delta &= \mathrm{d}^l \otimes (\partial_l + \Delta^\alpha_{\sigma\beta} f^\sigma_l \dot{y}^\beta \dot{\partial}_\alpha) \\ \Gamma^* &= \mathrm{d}^l \otimes (\partial_l - \Gamma^j_{lk} \dot{x}_j \dot{\partial}^k) \\ \otimes^i \Gamma^* \otimes f^*\Delta &= \mathrm{d}^j \otimes [\partial_j + (\Delta^\alpha_{\sigma\beta} f^\sigma_j w^\beta_{k_1 \dots k_i} - \Gamma^l_{jk} w^\alpha_{lk_2 \dots k_i} - \dots - \Gamma^l_{jk} w^\alpha_{k_1 \dots k_{i-1} l}) \partial_\alpha^{k_1 \dots k_i}]. \end{aligned}$$

Now we can introduce new definitions useful for the geometrical consideration.

Let $E^i = \otimes^i T^*M \otimes_M f^*TN$ be a vector bundle over M and let us consider the connection $\otimes^i\Gamma^* \otimes f^*\Delta$ on E^i. Let $s : M \to E^i$ be a section.

By the *covariant differential of s with respect to the connections Γ, Δ and a smooth mapping $f : M \to N$*, we mean a section of $T^*M \otimes E^i = \otimes^{i+1}T^*M \otimes f^*TN = E^{i+1}$ given by

$$\nabla^{(\otimes^i\Gamma^*\otimes f^*\Delta)}s(x)(\eta) = \mathrm{vpr} \circ [\mathrm{c} \circ j^1 s - (\otimes^i\Gamma^* \otimes f^*\Delta) \circ s](x)(\eta),$$

where $\mathrm{vpr} := \mathrm{pr}_2 \circ \mathrm{vl}^{-1} : VE^i \to E^i$ is the vertical projection, $\mathrm{vl} : (E^i \times_M E^i) \to VE^i$ is the canonical vector bundle isomorphism (vertical lift), and $\mathrm{c} : J^1E \hookrightarrow T^*M \otimes_M TE$ is the contact morphism. Section $x \to \nabla_\eta^{(\otimes^i\Gamma^*\otimes f^*\Delta)}s(x) = \nabla^{(\otimes^i\Gamma^*\otimes f^*\Delta)}s(x)(\eta)$ of E^i is called *covariant derivative of s along η with respect to the connections Γ, Δ and a smooth mapping f.*

Covariant differential may be viewed as a operator $C^\infty(E^i) \to C^\infty(E^{i+1})$.

Consider the tangent mapping $Tf : TM \to f^*TN$ as an element of $C^\infty E^1$. By a *covariant differential of order r of the tangent mapping Tf* we mean a mapping $[\nabla^{(\Gamma,\Delta,f)}]^r : C^\infty E^1 \to C^\infty E^{r+1}$ given by

$$[\nabla^{(\Gamma,\Delta,f)}]^r = \overbrace{\nabla^{(\otimes^r\Gamma^*\otimes f^*\Delta)} \circ \nabla^{(\otimes^{r-1}\Gamma^*\otimes f^*\Delta)} \ldots \circ \nabla^{(\Gamma^*\otimes f^*\Delta)}}^{r}.$$

The chart expression of the covariant differential can be obtained using the coordinate expressions of connections Δ, $f^*\Delta$, Γ, Γ^* and $\otimes^i\Gamma^* \otimes f^*\Delta$:

THEOREM 10. *In coordinates, covariant differential $[\nabla^{(\Gamma,\Delta,f)}]^k Tf$ has the following form:*

$$[\nabla^{(\Gamma,\Delta,f)}]^k Tf = K^\sigma_{i_1\ldots i_{k+1}} \partial_\sigma \otimes \mathrm{d}^{i_1} \otimes \mathrm{d}^{i_{k+1}},$$

where coefficients $K^\sigma_{i_1\ldots i_{k+1}}$ are given by the recurent formulae (2).

4. Physical application

In physical situations invariant Lagrangians can be constructed using the basis of differential invariants. For illustration we formulate some practically useful definitions and theorems only. For general aspects see [11], [13].

By $f : M \to N$ we denote again a smooth mapping of manifolds M, N with metric fields h, g, respectively.

A *k-th order invariant Lagrangian of f* is defined as an m-form on M

$$\Lambda = L(h_{ij}(x), \ldots, h_{ij,k_1\ldots k_r}(x), g_{\sigma\nu}(f(x)), \ldots, g_{\sigma\nu,\eta_1\ldots\eta_r}(f(x)),$$

$$f_i^\sigma(x), \ldots, f^\sigma_{i_1\ldots i_{r+1}}(x))\sqrt{|\det(h_{ij})|}\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m$$

such that $L : S^k \to \mathrm{R}$ is a $(G_m^{k+1} \times G_n^{k+1})$-invariant function. Invariant Lagrangians of smooth mapping are used in string theory, see e.g. [5]. The following property of invariant Lagrangians of smooth mappings can be derived as a consequence of Theorems 9, 10.

Corollary 11. *Any invariant k-th order Lagrangian of smooth mapping of manifolds with metric fields depends on metric fields, curvature tensors of the metric fields, and their covariant differentials up to the order $k-2$ and covariant differentials of the tangent mapping up to the order k.*

Remark: Analogously the problem of finding natural operators of smooth mappings of manifolds with connections can be formulated. In this case the basis of invariants is formed by coordinates R_{ijpl}, $R_{ijpl;m_1}$, …, $R_{ijpl;m_1;\dots;m_{k-2}}$, $\Omega_{\sigma\nu\eta\omega}$, $\Omega_{\sigma\nu\eta\omega;\gamma_1}$, …, $\Omega_{\sigma\nu\eta\omega;\gamma_1;\dots;\gamma_{k-2}}$, $K^{\sigma}_{i_1}$, …, $K^{\sigma}_{i_1\dots i_{k+1}}$. Consequently, there are no nontrivial Lagrangians of smooth mappings of manifolds with connections.

It is evident that the quotient space $S^k/(K^{k+1,1}_m \times K^{k+1,1}_n)$ is isomorphic with some Euclidean space with the tensorial action of the group $G^1_m \times G^1_n$. Then, using the methods of the classical theory of invariants (see [4], [20], [15]) it can be proved (see [19]):

Theorem 12. *Any algebraic $(G^1_m \times G^1_n)$-invariant function $L : S^k/(K^{k+1,1}_m \times K^{k+1,1}_n) \to$ R can be constructed from the basis of differential invariants by operations of the tensor product, alternation, contraction and lifting of indexes with respect to the metric.*

Thus, using this Theorem and knowing the basis of differential invariants (Theorem 9) we can completely solve the problem of construction of all algebraic invariant k-th order Lagrangians of smooth mapping of manifolds with metric fields.

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REFERENCES

[1] M. Atiyah, R. Bott, V. K. Patodi: *On the Heat Equation and the Index Theorem*, Inventiones Math. **19** (1973) 279—330, Errata **28** (1975) 277—280.

[2] J. Dieudonné: *Éléments d'Analyse*, Tome III, Gauthier-Villars, Paris 1974.

[3] P. B. Gilkey: *Curvature and the Eigenvalues of the Laplacian for Elliptic Complexes*, Advances in Math. **10** (1973) 344-382.

[4] G. B. Gurevich: *Foundation of the Theory of Algebraic Invariants*, Groningen 1964.

[5] P. Hořava, *Topological Rigid String Theory and Two Dimensional QCD*, Nucl. Phys. **B463** (1996) 238-286.

[6] J. Janyška, *Reduction Theorems for General Linear Connection*, Differential Geometry and Its Applications **20** (2004) 177-196.

[7] J. Janyška, *On the Curvature of Tensor Product Connections and Covariant Differentials*, Supplemento di Rendiconti del Circolo Matematico di Palermo, Serie II - No 72 - (2004) 135-143.

[8] J. Jost: *Riemannian Geometry and Geometric Analysis*, Second Edition, Springer-Verlag, Berlin 1998.

[9] I. Kolář, P. Michor, J. Slovák: *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin 1993.

[10] D. Krupka, J. Janyška: *Lectures on Differential Invariants*, J. E. Purkyně University, Brno 1990.

[11] D. Krupka, J. Musilová: *Calculus of Odd Base Forms on Differential Manifold*, Folia Fac. Sci. Nat. UJEP Brunensis, Physica, **24** (1983).

[12] D. Krupka: *Local Invariants of a Linear Connection*, Coll. Math. Soc, János Bolyai, 31. Differential Geometry, Budapest (Hungary) (1979); North Holland (1982) 349-369.

[13] D. Krupka: *Natural Lagrangian Structures*, Differential geometry Banach center publications, 12, Warsaw 1984.

[14] S. Lang:*Fundamentals of Differential Geometry*, Springer Verlag, Berlin 2001.

[15] P. Olver: *Classical Invariant Theory*, Cambridge University, Cambridge 1999.

[16] L. Mangiarotti, M. Modugno:*Connections and Differential Calculus on Fibred Manifolds. Applications to Field Theory.*, preprint Istituto di Mathematica Applicata "G. Sansone", Florence 1989.

[17] P. Musilová, J. Musilová: *Differential Invariants of Immersions of Manifolds with Metric Fields*, Communications in Mathematical Physics (2004), DOI: 10.1007/s00220-004-1113-8.

[18] P. Musilová, D. Krupka: *Differential Invariants of Immersions of Manifolds with Metric Fields*, Reports on Mathematical Physics**51** (2003)

[19] P. Musilová: *Differential Invariants of Immersions of Manifolds with Metric Fields*, Ph.D. Thesis, Masaryk University, Brno 2002.

[20] H. Weyl: *The Classical Groups, Their Invariants and Representations*, Princeton University, Princeton 1997.