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Controlling birefringence in dielectrics

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Birefringence, from the very essence of the word itself, refers to the splitting of light rays into two parts. In natural birefringent materials, this splitting is a beautiful phenomenon, resulting in the perception of a double image. In optical metamaterials, birefringence is often an unwanted side effect of forcing a device designed through transformation optics[1–6] to operate in dielectrics. One polarization is usually implemented in dielectrics, and the other is sacrificed[7,8]. Here we show, with techniques beyond transformation optics, that this need not be the case, that both polarizations can be controlled to perform useful tasks in dielectrics, and that rays, at all incident angles, can even follow different trajectories through a device and emerge together as if the birefringence did not exist at all. A number of examples are shown, including a combination Maxwell fisheye/Luneburg lens that performs a useful task and is achievable with current fabrication materials.

Transformation optics[1–6] has recently proven to be a useful methodology in the design of optical devices. Together with the theory of perfect lenses[9], it has become an important addition to the collection of mathematical optics tools available to researchers, useful in the study of invisibility cloaks[2,3,6,7,10–15], superabsorbers[16] and myriad other devices. One problematic feature of transformation optics, however, is that it requires the use of magnetic materials to retain full functionality in any device created. Consider a medium with an index of refraction n, where n is a function of space and not time. This can be represented in relative permittivity and permeability tensors (ε and μ, respectively) in spherical coordinates, with the diagonal tensor elements given by

$$\varepsilon = \{\varepsilon_r,\ \varepsilon_\theta,\ \varepsilon_\phi\} = \{n^2,\ n^2,\ n^2\},$$
$$\mu = \{\mu_r,\ \mu_\theta,\ \mu_\phi\} = \{1,\ 1,\ 1\} \tag{1}$$

If a radial optical transformation is applied to the tensors in equation (1) from original coordinate r (virtual space) to new coordinate R (physical space), then the tensors would take on new values as follows, with the optical device retaining complete functionality as the original isotropic index distribution. In the discussion that follows, $R(r)$ will be referred to as the transformation function.

$$\varepsilon = \left\{n^2\frac{r^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}r},\ n^2\frac{\mathrm{d}r}{\mathrm{d}R},\ n^2\frac{\mathrm{d}r}{\mathrm{d}R}\right\},\quad \mu = \left\{\frac{r^2}{R^2}\frac{\mathrm{d}R}{\mathrm{d}r},\ \frac{\mathrm{d}r}{\mathrm{d}R},\ \frac{\mathrm{d}r}{\mathrm{d}R}\right\} \tag{2}$$

It is clear from equation (2) that the resulting optical device is optically anisotropic and requires non-dielectric magnetic materials. This precludes fabrication for many devices. What has been done in the past to implement some devices in dielectrics is to recognize that the ray trajectories (and corresponding wave equations) of each polarization rely on different parameters. In any plane through the origin (which can be defined as the $r\phi$ plane in the spherical coordinate system) the in-plane-polarized ray trajectories depend only on ε_r, μ_θ and ε_ϕ, and the out-of-plane (θ-polarized) ray trajectories depend on μ_r, ε_θ and μ_ϕ. Maxwell's equations are rescalable within the regime of geometrical optics where it is assumed that the material parameters vary on longer scales than the wavelength. So, if each term in equation (2) is scaled by $\mathrm{d}r/\mathrm{d}R$ (multiplication for ε and division for μ), some of the values of μ become 1. If care is taken to match boundary conditions at the edge of any rescaled region, the in-plane polarization does not require magnetic materials because $\mu_\theta = 1$, relying only on the boxed terms in equation (3):

$$\varepsilon = \left\{\boxed{n^2\frac{r^2}{R^2}},\ n^2\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2,\ \boxed{n^2\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2}\right\},\quad \mu = \left\{\frac{r^2}{R^2}\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2,\ \boxed{1},\ 1\right\} \tag{3}$$

If μ_r is also forced to be 1, then the device can be implemented in dielectrics, but correct operation of the (unboxed) out-of-plane polarization is sacrificed. This rescaling operation is how the microwave cloaking device[7] and the Eaton lens[8], for example, were designed and fabricated for operation with only one polarization intact, although in those cases the rescaling was not for $\mu = 1$, because operation was in the microwave regime. Here, we examine the behaviour of the sacrificed polarization to see how its behaviour can be controlled. At first glance, the situation appears to be hopeless, because in equation (3), if we assign the term $\mu_r = (r^2/R^2)(\mathrm{d}R/\mathrm{d}r)^2 = 1$, then it forces the transformation function to be $R(r) = Cr$ (where C is a constant) and nothing is gained. However, this is incorrect, because we are not forced to apply the methods of transformation optics to both polarizations. In fact, the two polarizations can be designed independently. If we simply assign $\mu_r = 1$, it need not have any bearing on the transformation function, as we are free to choose μ_r, ε_θ and μ_ϕ to take any values without affecting the in-plane polarization's ray trajectories through the device. Choosing all values of μ to be 1 and choosing $\varepsilon_\theta = \varepsilon_\phi$ to retain spherical symmetry results in the following expression for the tensor diagonals:

$$\varepsilon = \left\{n_r^2 = n^2\frac{r^2}{R^2},\ n_\theta^2 = n^2\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2,\ n_\phi^2 = n^2\left(\frac{\mathrm{d}r}{\mathrm{d}R}\right)^2\right\},$$
$$\mu = \{1, 1, 1\} \tag{4}$$

Like the rescaling done in equation (3), the replacement in equation (4) works perfectly for waves in the regime of geometrical optics. The in-plane polarization has been designed through transformation optics and behaves functionally like the isotropic medium n, even though the rays may follow different trajectories depending on the choice of $R(r)$. The out-of-plane polarization behaves functionally like an isotropic medium of index n_θ. However, to retain spherical symmetry, the relationship $n_\theta^2 = n^2(\mathrm{d}r/\mathrm{d}R)^2$ from equation (4) restricts the transformation function and also restricts device design in ways that will be discussed below. If the physical

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extent of the device is at $R = r = 1$, then solving the following first-order differential equation will yield the correct transformation function $R(r)$:

$$n_\theta(R)\frac{\mathrm{d}R}{\mathrm{d}r} = n(r), \quad R\big|_{r=0} = R_0 \geq 0, \quad R\big|_{r=1} = 1, \quad R(r) \text{ not multivalued} \tag{5}$$

This can be easily integrated as $\int_{R_0}^{R} n_\theta(R)\mathrm{d}R = \int_0^r n(r)\mathrm{d}r$. Equation (5) describes the design methodology that must be used if dielectric devices with compatible birefringence are to be designed. Although simple, this solves a long-standing problem in optics and the consequent restrictions on device design are the reason why Eikonal-like solutions for the in-plane polarization were never found as they were in the isotropic case[17–19]. It is important to point out, however, that although dielectrics have not been considered until now, the independent choosing of two sets of medium parameters to realize different optical functions has been previously reported[20]. A number of consequences result from this theory. First, it has been proven that cloaking is impossible with spherically symmetric birefringent dielectrics. This is because the out-of-plane polarization along any plane passing through the origin cannot have functionality beyond that of an isotropic material (its ray trajectories depending only on a single index n_θ), and impossibility of cloaking has previously been shown for spherically symmetric isotropic materials[21]. Second, complete freedom is available in designing devices that have multiple functions. Although nothing is gained from birefringence in terms of functionality for the out-of-plane polarization, for the in-plane polarization cloaking is possible (for example), and all advantages of transformation optics are still present. Rather than sacrificing the out-of-plane polarization when implementing a device designed with transformation in dielectrics, use of this theory can give functionality to both polarizations. Third, not all combinations of functionality in birefringent devices are possible. For example, if one wanted a device to function like free space for out-of-plane polarization (with $n_\theta = 1$) but like an Eaton lens with $n(r) = \sqrt{(2/r - 1)}$ for the other polarization, this would be impossible within the framework of equation (5). If the polarizations were reversed, then it is possible by numerically finding a radius R_0 and then solving equation (5) for the transformation function $R(r)$. The 'stronger' index must be implemented with the out-of-plane polarization. If the integral of an index is too small, however, additional loops around the origin can be added to the ray trajectories to increase it at the expense of having singularities in the resulting profile. This will be illustrated next.

To show the utility of the method, example devices were designed (Fig. 1; see Supplementary Information). Figure 1 shows ray traces of light trajectories in several lenses that will be described. Ray tracing was carried out numerically, independently of lens design, using procedures described in detail in refs 22 and 23. The resultant ray diagrams, both here and in the Supplementary Information, thus serve as numerical confirmation of the validity of the theory.

In Fig 1a, the goal was to design a lens that behaves like free space for the out-of-plane polarization and like a Luneburg lens[17] ($n(r) = \sqrt{(2 - r^2)}$) for the other. A Luneburg lens is a spherically symmetric lens that focuses plane waves at its rim. Because free space cannot be matched to the out-of-plane polarization for reasons just described, an additional loop was inserted into all ray trajectories, creating an invisible sphere[24] with index given by

$$n_\theta = \left(Q - \frac{1}{3Q}\right)^2, \quad Q = \sqrt[3]{-\frac{1}{r} + \sqrt{\frac{1}{r^2} + \frac{1}{27}}} \tag{6}$$

The constant in equation (5) that accomplishes this is $R_0 = 0.2556$. A photorealistic representation of this device's behaviour is shown in the supplementary picture. Note that the index $n(r) = \sqrt{(2 - r^2)}$ describing the Luneburg behaviour is the index before transformation, as the index n_r (R) after transformation can only be expressed numerically.

In Fig. 1b, the goal was to have a Luneburg lens for the out-of-plane polarization ($n_\theta(R) = \sqrt{(2 - R^2)}$), and a ring focuser (all rays entering a plane will be focused into a ring, with profile $n = r$) for the in-plane polarization. This is accomplished with $R_0 = 0.5713$.

Figure 1c shows an invisible sphere, but with birefringence not apparent at the exterior of the device. To the best of our knowledge, this is the first time it has been shown to be possible for light rays to enter a device from any direction, split and follow different trajectories, and then recombine. The only evidence of the birefringence at the exterior of the device would be a polarization-dependent phase slip, because light for the out-of-plane polarization follows a longer optical path. The index of the invisible sphere is again given by equation (6), and the transformed index is $n = 1$, with $R_0 = 0.3544$.

Another example is shown in Fig. 1d, and is a device that could function as an illumination lens for a Maxwell fisheye. A Maxwell fisheye is a perfect lens with a resolution not limited by the wave nature of light[25,26], but it is an unusual 'lens', with both source and image lying inside it. Our method can be used to illuminate a fisheye lens from within using a quantum dot. The power to the quantum dot could be delivered by an external pump (having passed through a special polarizer), which could be focused by a gradient index lens profile to the quantum dot within the fisheye. When the quantum dot re-emits, some of the light would be in the opposite polarization, and a dichroic mirror or polarization-dependent mirror placed around the fisheye could capture this light as in Fig. 1d, while allowing the pump light to pass. We thus need a device that functions as a Maxwell fisheye for one polarization and a gradient index interior-focusing lens for the other. A Maxwell fisheye profile is assigned to the in-plane polarization with $n(r) = 2/(1 + r^2)$. To avoid singularities in index n_r after transformation, constant $R_0 = 0$. This places significant restriction on the choice for n_θ and the functionality for the out-of-plane polarization. Solving equation (5) with the fisheye profile and $R_0 = 0$ yields $\int_0^1 n(r)\mathrm{d}r = \pi/2$, which restricts our choice of n_θ to potentials that satisfy $\int_0^1 n_\theta(R)\mathrm{d}R = \pi/2$. We have developed new gradient index lens design methods for this purpose. As explained in detail in the Supplementary Information, a potential n_θ was found that focuses plane waves to a point inside the sphere (at radius $R = 0.6621$ in this particular example). It is noted that the index requirements on the new device are actually reduced relative to a standard Maxwell fisheye. At the origin, the birefringence disappears.

Further examples of the use of this theory are described in the Supplementary Information. Specifically, there exists a class of functions that can satisfy the constraint of no index singularities while still allowing independent control over the functionality of a lens for each polarization. This is possible by requiring that $n_\theta = 1$ for greater than some radius, which can be adjusted to give $R_0 = 0$ in equation (5) and hence avoid singularities at the expense of lens aperture for one polarization. Essentially, a lens can have different effective diameters for each polarization, which gives another means of control.

Transformation optics allows the design of useful gradient index devices, but implementation in dielectrics is not possible for both polarizations because of the requirement that $\mu = \varepsilon$ after transformation, which cannot be fixed with simple rescaling. We have shown that optical transformation theory need only apply to one polarization, and that useful optical functions can still be implemented with the other polarization. In fact, this study was originally motivated by invisibility cloaks. Because all reported three-dimensional cloaks require a non-unity magnetic permeability, they are out of reach of dielectrics unless one polarization is sacrificed. This

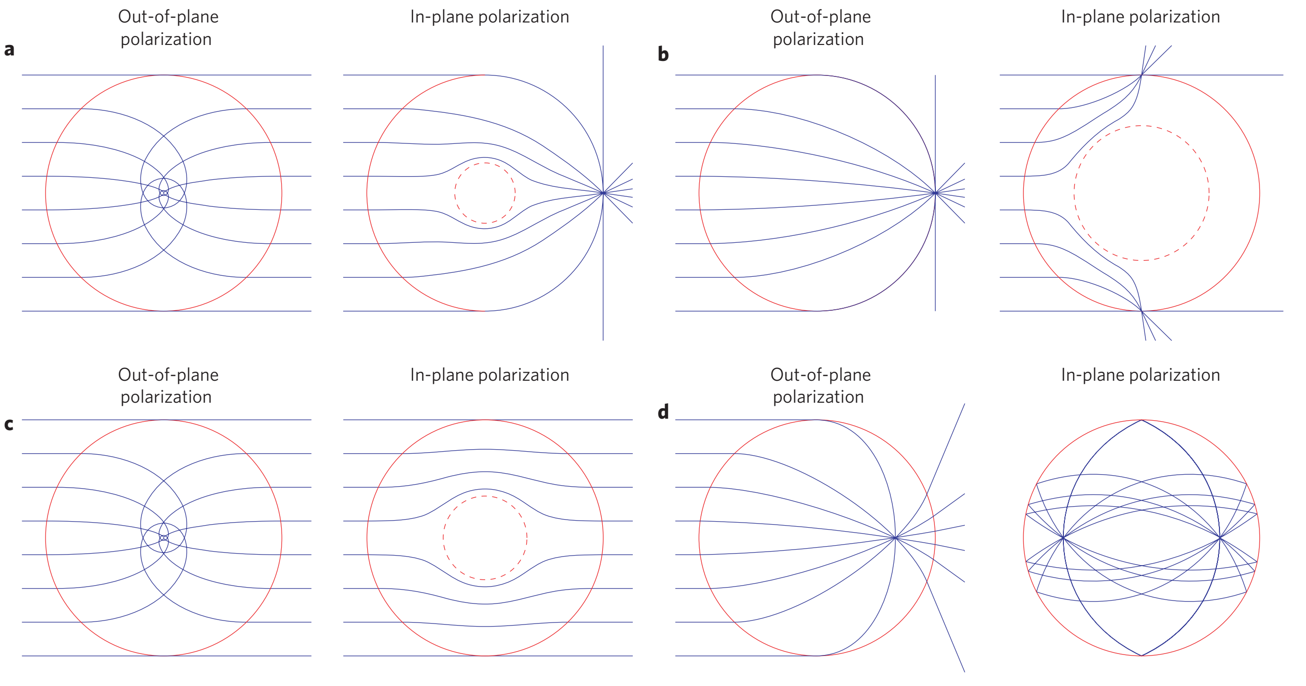

Figure 1 | Four examples of birefringent dielectric devices designed with the methods described. a, Invisible for one polarization, a Luneburg lens for the other as shown in the supplementary picture. **b**, A Luneburg lens for one polarization, and a ring focuser for the other. **c**, Invisible but with a polarization-dependent phase slip. **d**, An interior-focusing potential for one polarization and a Maxwell fisheye for the other.

theory has shown that this may not be necessary and is therefore significant. Even if a cloak would ultimately work only for one polarization, at least the other polarization could do something useful (divert rays to camouflage, for example). With the methods developed herein, birefringence can be fully controlled in dielectrics, and useful optical devices can be designed that have either multiple functions or identical functionality for each polarization but with different ray trajectories. The gradient interior focusing problem has also been solved and has been used in combination with the theory of birefringence to design a Maxwell fisheye/Luneburg combination lens that is within reach of materials suitable for fabrication.

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Author contributions

A.D. devised the main theory presented in the text and created ray-traced images. T.T. contributed to designing the interior focusing and multifocal length lenses. U.L. proposed the original problem and contributed to potential applications of the theory.

Additional information

The authors declare no competing financial interests. Supplementary information accompanies this paper at www.nature.com/naturephotonics. Reprints and permission information is available online at http://www.nature.com/reprints/. Correspondence and requests for materials should be addressed to A.J.D.